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NOTE ON A NONLINEAR VOLTERRA EQUATION

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1. Introduction.

We investigate the solutions of

$$x'(t) = -\int_0^t a(t-\tau)g(x(\tau))d\tau \qquad \left(' = \frac{d}{dt}\right) \tag{1.1}$$

as $t \to \infty$, where $a(t)$ is completely monotonic on $0 \leqq t < \infty$ and where $g(x)$ is a (nonlinear) spring. Under this hypothesis, (1.1) was shown in [2] to be relevant to certain physical applications and results were obtained there for the linear case $g(x) \equiv x$. (If $a(t) \equiv a(0)$, then (1.1) reduces to the nonlinear oscillator $x'' + a(0)g(x) = 0$.) Equation (1.1) was studied in [1] under less hypothesis on $a(t)$. However, while the result is weaker than that of [1], the present approach draws together such different notions of positivity as Liapounov functions, completely monotonic functions, and kernels of positive type. It also provides a new Liapounov function for (1.1). Specifically, we prove the

THEOREM. *Let $a(t)$ and $g(x)$ satisfy*

$$a(t) \in C[0, \infty), \ (-1)^k a^{(k)}(t) \geqq 0 \ (0 < t < \infty; k = 0, 1, 2, \cdots), \tag{1.2}$$

$$g(x) \in C(-\infty, \infty), \ xg(x) > 0 \ (x \neq 0), \ G(x) = \int_0^x g(\xi)d\xi \to \infty \quad (|x| \to \infty). \tag{1.3}$$

Presented to the Society, January 24, 1963; received by the editors September 6, 1962.

[1] MIT Lincoln Laboratory. Operated with support from the U. S. Army, Navy, and Air Force.

[2] Partially supported by the National Science Foundation (G-19925).

If $a(t)\not\equiv a(0)$ *and if* $u(t)$ *is any solution of* (1.1) *which exists on* $0\leqq t<\infty$, *then*

$$\lim_{t\to\infty} u^{(j)}(t) = 0 \qquad (j = 0, 1, 2). \tag{1.4}$$

In [1] only $k=0, 1, 2, 3$ is required in the analogue of (1.2), rather than complete monotonicity. The Liapounov function used there was

$$\begin{aligned} E(t) = G(u(t)) &+ \frac{1}{2}\, a(t)\left[\int_0^t g(u(\tau))d\tau\right]^2 \\ &- \frac{1}{2}\int_0^t a'(t-\tau)\left[\int_\tau^t g(u(s))ds\right]^2 d\tau \geqq 0. \end{aligned} \tag{1.5}$$

In (1.5) and the sequel $u(t)$ is the solution of (1.1) on $0\leqq t<\infty$ being considered. Remarks concerning existence, uniqueness, as well as background information and references, may be found in [1].

2. **Positivity.** In this section we motivate the hypothesis (1.2) and also obtain the Liapounov function. Suppose that the origin of the problem is such that $a(t)\geqq 0$, $a(t)\in C[0, \infty)$, $a'(t)\in L_1(0, T)$ for each $0<T<\infty$, and (1.3) are all required. Then clearly

$$V(t) = G(u(t)) + \frac{1}{2}\int_0^t\int_0^t a(\tau+s)g(u(t-\tau))g(u(t-s))d\tau ds \tag{2.1}$$

is nonnegative if the second term is. (From (1.1) and (1.3), $V(t)$ may be interpreted as the sum of a potential and kinetic energy.)

The second (or kinetic) term of $V(t)$ will be nonnegative if it is assumed that $a(\tau+s)$ is a kernel of positive type [3, p. 270] on the square $0<\tau, s<t$ for each $0<t<\infty$. By a theorem of Boas and Widder [3, pp. 273–275], this will be the case if and only if

$$a(t) = \int_{-\infty}^{\infty} \exp[-\xi t]d\alpha(\xi) \qquad (0<t<\infty), \tag{2.2}$$

where $\alpha(\xi)$ is nondecreasing on $-\infty<\xi<\infty$ (and may be assumed normalized: $\alpha(0)=0$, $\alpha(\xi)=\frac{1}{2}[\alpha(\xi+)+\alpha(\xi-)]$).

Differentiating (2.1) yields

$$V'(t) = \int_0^t\int_0^t a'(\tau+s)g(u(t-\tau))g(u(t-s))d\tau ds, \tag{2.3}$$

where we have used the above assumptions, (1.1), the absolutely continuity of $a(t)$ on $0<\epsilon\leqq t\leqq T<\infty$ implied by (2.2), and an obvious change of variables. If $V(t)$ is to serve as a Liapounov function for

(1.1), then $V'(t)$ must be nonpositive. By (2.3) the latter will be assured if $-a'(\tau+s)$ is a kernel of positive type on the square $0<\tau$, $s<t$ for each $0<t<\infty$. However, this is compatible with (2.2) if and only if $\alpha(-\infty)=\alpha(0-)$. Thus, also using $a(t)\in C[0, \infty)$, one has

$$a(t) = \int_0^\infty \exp[-\xi t]d\alpha(\xi) \qquad (0 \leqq t < \infty), \tag{2.4}$$

where $\alpha(\infty)<\infty$. By a theorem of S. Bernstein [**3**, p. 160], (1.2) and (2.4) are equivalent.

Having motivated the hypothesis (1.2) and obtained a Liapounov function $V(t)$, it is interesting to compare the two Liapounov functions (1.5) and (2.1). It is obvious that if $a(t)\equiv a(0)$, then $E(t) \equiv V(t)$ $(0\leqq t<\infty)$. We now prove a strong form of the converse statement. In particular, if $u(0)\neq 0$ and if $E(t)\equiv V(t)$ $(0\leqq t\leqq t_0)$ for some $0<t_0<\infty$, then $a(t)\equiv a(0)$ $(0\leqq t<\infty)$. For this one need only assume that $a(t)\in C[0, \infty)$, $(-1)^k a^{(k)}(t)\geqq 0$ $(0<t<\infty;\ k=0, 1, 2)$, $g(x)\in C(-\infty, \infty)$, and $g(x)=0$ implies $x=0$. Direct calculations show that (1.5) and (2.1) may also be written as

$$E(t) = G(u(t)) + \int_0^t g(u(\tau))\left\{\int_\tau^t a(t - \tau)g(u(s))ds\right\}d\tau,$$

$$V(t) = G(u(t)) + \int_0^t g(u(\tau))\left\{\int_\tau^t a(2t - \tau - s)g(u(s))ds\right\}d\tau.$$

Hence

$$\int_0^t g(u(\tau))\left\{\int_\tau^t [a(t - \tau) - a(2t - \tau - s)]g(u(s))ds\right\}d\tau \equiv 0 \qquad (0 \leqq t \leqq t_0). \tag{2.5}$$

From the hypothesis on $a(t)$ one has $a(t-\tau)-a(2t-\tau-s)\geqq 0$ for $0\leqq\tau$, $s\leqq t$. As $u(0)\neq 0$, there exists by continuity and the hypothesis on $g(x)$ a $0<t_1\leqq t_0$ such that $g(u(t))\neq 0$ for $0\leqq t\leqq t_1$. It is now clear from (2.5) that $a(t-\tau)\equiv a(2t-\tau-s)$ $(0\leqq\tau\leqq s\leqq t\leqq t_1)$. The latter together with $a(t)\in C[0, \infty)$ easily implies $a(t)\equiv a(0)$ $(0\leqq t\leqq 2t_1)$. Hence $a'(t)\equiv 0$ $(0\leqq t\leqq 2t_1)$. However, as $-a'(t)$, $a''(t)\geqq 0$ it follows that $a'(t)\equiv 0$ $(0\leqq t<\infty)$ so that $a(t)\equiv a(0)$ $(0\leqq t<\infty)$, which proves the assertion.

3. Proof of the Theorem. Let $a(t)$, $V(t)$ be given by (2.4), (2.1) respectively. Define

$$(3.1)\qquad \Gamma(\xi, t) = \int_0^t \exp[-\xi(t-\tau)]g(u(\tau))d\tau \quad (0 \leqq \xi, t < \infty).$$

Then

$$(3.2)\qquad \frac{\partial \Gamma}{\partial t}(\xi, t) = g(u(t)) - \xi\Gamma(\xi, t) \qquad (0 \leqq \xi, t < \infty).$$

Clearly $\Gamma(\xi, t)$, $\Gamma_t(\xi, t)$ are bounded functions of ξ on $0 \leqq \xi < \infty$ for each fixed t in $0 \leqq t < \infty$. Hence, by Fubini's theorem, one may write (1.1) (with $u(t)$ replacing $x(t)$), (2.1) as

$$(3.3)\qquad u'(t) = -\int_0^\infty \Gamma(\xi, t)d\alpha(\xi) \qquad (0 \leqq t < \infty),$$

$$(3.4)\qquad V(t) = G(u(t)) + \frac{1}{2}\int_0^\infty \Gamma^2(\xi, t)d\alpha(\xi) \geqq 0 \qquad (0 \leqq t < \infty)$$

respectively and, moreover,

$$(3.5)\qquad V'(t) = -\int_0^\infty \xi\Gamma^2(\xi, t)d\alpha(\xi) \leqq 0 \qquad (0 \leqq t < \infty).$$

From (3.4), (3.5) one has

$$G(u(t)) \leqq V(t) \leqq V(0) = G(u_0) \qquad (0 \leqq t < \infty),$$

where $u_0 = u(0)$. It follows from (1.3) that

$$(3.6)\qquad |u(t)| \leqq K < \infty \qquad (0 \leqq t < \infty).$$

In (3.6) and subsequent formulas $K = K(u_0) < \infty$, where K may vary from formula to formula, and $K(u_0) \to 0$ as $u_0 \to 0$. Thus

$$(3.7)\qquad |\Gamma(\xi, t)| \leqq Kt, \quad |\xi\Gamma(\xi, t)| \leqq K, \quad |\Gamma_t(\xi, t)| \leqq K \quad (0 \leqq \xi, t < \infty).$$

Differentiating (3.3) (using Fubini's theorem) yields

$$(3.8)\qquad u''(t) = -g(u(t))\alpha(\infty) + \int_0^\infty \xi\Gamma(\xi, t)d\alpha(\xi) \qquad (0 \leqq t < \infty),$$

which together with (1.3), (3.6), and (3.7) implies

$$(3.9)\qquad |u''(t)| \leqq K \qquad (0 \leqq t < \infty).$$

By (3.6), (3.9), and the mean value theorem one has

$$(3.10)\qquad |u'(t)| \leqq K \qquad (0 \leqq t < \infty).$$

From (3.2), (3.5) there results

$$V''(t) = -2g(u(t))\int_0^\infty \xi\Gamma(\xi, t)d\alpha(\xi) + 2\int_0^\infty \xi^2\Gamma^2(\xi, t)d\alpha(\xi) \qquad (0 \leqq t < \infty),$$

so that by (1.3), (3.6), (3.7) one has $|V''(t)| \leqq K$ on $0 \leqq t < \infty$. The latter together with (3.4), (3.5), and the mean value theorem implies (see Lemma 1 of [1])

$$\lim_{t\to\infty} V'(t) = -\lim_{t\to\infty}\int_0^\infty \xi\Gamma^2(\xi, t)d\alpha(\xi) = 0. \tag{3.11}$$

We assert that there exists a $\xi_1 > 0$ such that $\Gamma(\xi_1, t) \to 0$ as $t \to \infty$. Suppose not and let $0 < \xi_0 < \infty$. Then there exist a $\lambda = \lambda(\xi_0) > 0$ and a sequence $\{t_n = t_n(\xi_0)\}$, where $t_n \to \infty$ as $n \to \infty$, such that $|\Gamma(\xi_0, t_n)| \geqq \lambda$. From (3.1) one has

$$\frac{\partial\Gamma}{\partial\xi}(\xi, t) = -\int_0^t \exp[-\xi(t-\tau)](t-\tau)g(u(\tau))d\tau \qquad (0 \leqq \xi, t < \infty). \tag{3.12}$$

Let $\delta = \min(\xi_0/2, \lambda\xi_0^2/8K_1)$, $I_{\xi_0} = \{\xi \mid |\xi - \xi_0| \leqq \delta\}$, where $|g(u(t))| \leqq K_1$ on $0 \leqq t < \infty$. Then by the mean value theorem and (3.12) one obtains $|\Gamma(\xi, t_n) - \Gamma(\xi_0, t_n)| \leqq \lambda/2$ $(\xi \in I_{\xi_0})$ so that, as $|\Gamma(\xi_0, t_n)| \geqq \lambda$, $|\Gamma(\xi, t_n)| \geqq \lambda/2$ $(\xi \in I_{\xi_0})$. Hence

$$\begin{aligned}\int_0^\infty \xi\Gamma^2(\xi, t_n)d\alpha(\xi) &\geqq \int_{I_{\xi_0}} \xi\Gamma^2(\xi, t_n)d\alpha(\xi) \\ &\geqq \frac{\lambda^2\xi_0}{8}[\alpha(\xi_0 + \delta) - \alpha(\xi_0 - \delta)] \geqq 0,\end{aligned}$$

which with (3.11) yields $\alpha(\xi_0 + \delta) = \alpha(\xi_0 - \delta)$. As this is true for each $0 < \xi_0 < \infty$, it follows that $\alpha(0+) = \alpha(\infty)$ which contradicts $a(t) \not\equiv a(0)$. Thus, there exists a $\xi_1 > 0$ with the asserted property.

Let $f(t) = \exp[-\xi_1 t]$, $p(t) = g(u(t))$ for $0 \leqq t < \infty$ and $f(t) = p(t) \equiv 0$ for $-\infty < t < 0$. From the preceding paragraph one has

$$\lim_{t\to\infty} \Gamma(\xi_1, t) = \lim_{t\to\infty}\int_{-\infty}^\infty f(t-\tau)p(\tau)d\tau = 0. \tag{3.13}$$

By applying Pitt's form of Wiener's tauberian theorem [**3**, p. 211] to (3.13), we now show that $p(t) \to 0$ as $t \to \infty$. (A longer elementary argument could also be used here.) Clearly, $f \in L_1(-\infty, \infty)$ and its Fourier transform $\hat{f}(s) = (2\pi)^{-1/2}(\xi_1 + is)^{-1} \neq 0$ for $-\infty < s < \infty$. As $|p(t)| \leqq K$ $(-\infty < t < \infty)$, there remains only to show that $p(t)$ is a slowly decreasing function in $(-\infty, \infty)$. For this it suffices to show that $p(t_i)-p(s_i)\to 0$ as $i\to\infty$ if $\{t_i\}$, $\{s_i\}$ are any sequences satisfying $t_i>s_i>0$ and $s_i\to\infty$, $t_i-s_i\to 0$ as $i\to\infty$. However, from (1.3), (3.6), (3.10), the mean value theorem, and uniform continuity, it is clear that $g(u(t_i))-g(u(s_i))\to 0$ as $i\to\infty$ for such sequences. Thus,[3] $p(t)\to 0$ as $t\to\infty$, which together with (1.3) and (3.6) yields $u(t)\to 0$ as $t\to\infty$ (i.e., $(1.4, j=0)$).

From $(1.4, j=0)$ and (3.9) one has $(1.4, j=1)$ by the mean value theorem. From $(1.4, j=0)$ and (3.1) it is an elementary exercise to show that $\xi\Gamma(\xi, t)\to 0$ as $t\to\infty$ uniformly with respect to ξ on $0\leqq\xi<\infty$. This together with $(1.4, j=0)$, (1.3), and (3.8) implies $(1.4, j=2)$, which completes the proof.

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