REMARKS ON SYSTEMS OF NONLINEAR VOLTERRA EQUATIONS

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Levin [1], and Levin and Nohel [2] have investigated the asymptotic behaviour of the solutions of the nonlinear integro-differential equation

$$(1) \qquad x'(t) = - \int_0^t a(t - \tau)g(x(\tau))d\tau.$$

Here we shall indicate how both the results and the techniques of these papers can be generalized to the case of a system.

Thus we shall consider x and g to be vectors in E^n, regarded as $n \times 1$ matrices, and $a(t)$ will be an $n \times n$ matrix. In [3] Nohel has dealt with the questions of existence, uniqueness, and continuation of solutions for this case so we shall not dwell on that aspect.

In [2] the following assumptions are made for $n = 1$:

(2) $g(x) \in C(-\infty, \infty)$, $xg(x) > 0$ $(x \neq 0)$, $G(x) = \int_0^x g(\xi)d\xi \to \infty$ as $|x| \to \infty$, and

(3) $a(t) \in C[0, \infty)$, $(-1)^k a^{(k)}(t) \geq 0$ $(0 < t < \infty$; $k = 0, 1, 2, \cdots)$; while in [1] (3) is replaced by the weaker condition

(3') $a(t) \in C[0, \infty)$, $(-1)^k a^{(k)}(t) \geq 0$ $(0 < t < \infty$; $k = 0, 1, 2, 3)$. The result in both cases is the

THEOREM. *Any solution $u(t)$ of (1) satisfies*

$$(4) \qquad \lim_{t \to \infty} u^{(j)}(t) = 0 \qquad (j = 0, 1, 2),$$

provided only that $a(t) \not\equiv a(0)$.

The techniques used involve the construction of Liapounov functionals $E(t)$ in [1] and $V(t)$ in [2], and the generalization of the results to $n > 1$ is based on a straightforward generalization of these functionals.

For these generalizations we shall make the assumptions:

(5) $g \in C(E^n)$, $x^T g(x) > 0$ $(x \neq 0)$ (where x^T is the transpose of the $n \times 1$ matrix x), there exists a scalar function $G \in C^1(E^n)$ such that g is the gradient of G and $G(x) \to \infty$ as $|x| \to \infty$; and

(6) $a(t) \in C[0, \infty)$, and $(-1)^k a^{(k)}(t)$ is a real symmetric positive semi-definite matrix for $0 < t < \infty$, $k = 0, 1, 2, \cdots$, or

(6′) $a(t) \in C[0, \infty)$, and $(-1)^k a^{(k)}(t)$ is a real symmetric positive semi-definite matrix for $0 < t < \infty$, $k = 0, 1, 2, 3$.

The assumption (5) that g is a vector field possessing a potential is rather restrictive, but as it is automatically fulfilled for $n = 1$ it seems a reasonable first assumption to make in treating systems. Assumptions (6) and (6′) are immediate n-dimensional analogues of (3) and (3′) respectively. On the basis of these assumptions one proves the theorems stated above.

Using (5) and (6) the proof is based upon the Liapounov functional

$$(7) \quad V(t) = G(u(t)) + \frac{1}{2} \int_0^t \int_0^t g^T(u(t - \tau)) a(\tau + s) g(u(t - s)) ds d\tau$$

and using (5) and (6′) it is based on

$$
\begin{aligned}
(8) \quad E(t) = {} & G(u(t)) + \frac{1}{2} \left[\int_0^t g^T(u(\tau)) d\tau \right] a(t) \left[\int_0^t g(u(\tau)) d\tau \right] \\
& - \frac{1}{2} \int_0^t \left[\int_\tau^t g^T(u(s)) ds \right] a'(t - \tau) \left[\int_\tau^t g(u(s)) ds \right] d\tau.
\end{aligned}
$$

The details of the proofs involve the generalizations of the lemmas of [1] and [2] to apply to matrix and vector functions as well as to scalar functions. This involves some labour, but no essential difficulty.

REFERENCES

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3. J. A. Nohel, *Some problems in nonlinear Volterra integral equations*, Bull. Amer. Math. Soc. **68** (1962), 323–330.

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