HOMOTOPIC CURVES ON SURFACES

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In the following, a surface is always a compact, orientable, two-dimensional manifold with or without boundary, a closed surface is one without boundary, and a simple, closed curve on a surface is a closed, connected, one-dimensional submanifold. For simplicity in referring to curves, no notational distinction is made between the embedding of the circle and the image under the embedding—the context will make it clear which is intended.

If W is a surface whose boundary consists of n disjoint, simple, closed curves, the genus of W, $g(W)$, is defined to be the genus of a surface without boundary obtained by attaching a 2-cell to each of the boundary curves. The relation between the genus of W and its inner Euler characteristic, $\chi(W)$ is:

$$\chi(W) = 2 - 2g - n.$$

Any two surfaces with the same number of boundary curves and genus are homeomorphic [2], and if everything is differentiable, diffeomorphic [1]. A surface with one boundary curve and genus zero will be called a disc, and a surface with two boundary curves and genus zero will be called a cylinder.

No distinction is made between the topological and differentiable cases, since everything below applies equally well to both.

The object of this note is to prove:

THEOREM. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be simple, closed, nonintersecting curves on a closed surface, V, then:*

(i) *If $\mathfrak{A}$ is homotopic to zero, then $\mathfrak{A}$ bounds a disc in V.*

(ii) *If $\mathfrak{A}$ and $\mathfrak{B}$ are freely homotopic but not homotopic to zero, then $\mathfrak{A} \cup \mathfrak{B}$ bounds a cylinder in V.*

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1. LEMMA. *Suppose $a_1, \cdots, a_n$ are simple, closed, nonintersecting, null-homologous curves in V, then $V - \bigcup a_i$ has $(n+1)$ components.*

PROOF. Consider the cohomology sequence of the pair (V, A), where

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$A = \bigcup a_i$:

$$\cdots \to H^1(V) \xrightarrow{i^*} H^1(A) \to H^2(V, A) \to H^2(V) \to H^2(A) \to \cdots.$$

Since all of the a_i are null-homologous i^* is the zero map. $H^1(A) \approx Z^n$, $H^2(A) = 0$, and $H^2(V) \approx Z$. Thus we have:

$$0 \to Z^n \to H^2(V, A) \to Z \to 0.$$

So $H^2(V, A) \approx Z^{n+1}$, and the result follows by duality, $H^2(V, A) \approx H_0(V - A)$.

LEMMA. *Suppose $a_1, \cdots, a_n$ are simple, closed nonintersecting, homologous, but not null-homologous curves on V, then $V - \bigcup a_i$ has n components.*

PROOF. This proof is similar to the preceding one except that now i^* maps $H^1(V)$ onto the diagonal of Z^n. The resulting exact sequence is:

$$0 \to Z \to Z^n \to H^2(V, A) \to Z \to 0,$$

and so $H^2(V, A) \approx H_0(V - A) \approx Z^n$.

2. The theorem breaks up into three cases: (i) $\mathfrak{A}$ homotopic to zero, $(\text{ii})_1$ $\mathfrak{A}$ and $\mathfrak{B}$ freely homotopic, homologous to zero but not homotopic to zero, $(\text{ii})_2$ $\mathfrak{A}$ and $\mathfrak{B}$ freely homotopic but not homologous to zero.

In (i), since $\mathfrak{A}$ is homologous to zero the surface is divided by $\mathfrak{A}$ into two surfaces with genera g' and g'' and the assertion of the theorem is that g' or g'' is zero. Since the genus of the whole surface, $g = g' + g''$, we see that the result is immediate if $g \leqq 1$.

In $(\text{ii})_1$, since the curves $\mathfrak{A}$ and $\mathfrak{B}$ are null-homologous, V is divided by $\mathfrak{A}$ and $\mathfrak{B}$ into three surfaces: two with single bounding curves with genera g' and g''' and one with two boundary curves with genus g''. Since neither $\mathfrak{A}$ nor $\mathfrak{B}$ is null-homotopic, g' and g''' are nonzero. The assertion of $(\text{ii})_1$ is that g'' is zero. Since the genus of V, $g = g' + g'' + g'''$, the conclusion of $(\text{ii})_1$ holds for surfaces of genus $g \leqq 2$ if $\mathfrak{A}$ and $\mathfrak{B}$ are null-homologous and not null-homotopic (which is only possible if $g = 2$).

In $(\text{ii})_2$ since the curves $\mathfrak{A}$ and $\mathfrak{B}$ are homologous but not null-homologous, V is divided by $\mathfrak{A}$ and $\mathfrak{B}$ into two surfaces with two boundary curves each and genera g' and g''. The assertion of $(\text{ii})_2$ is that either g' or g'' vanishes. Since the genus of V, $g = g' + g'' + 1$, the conclusion of $(\text{ii})_2$ follows if $g \leqq 2$. Thus:

$(\text{ii})'$ *If V is a closed surface of genus $g \leqq 2$ and if $\mathfrak{A}$ and $\mathfrak{B}$ are simple,* *closed, nonintersecting curves which are homologous but not homotopic to zero in* V, *then* $\mathfrak{A}\cup\mathfrak{B}$ *is the boundary of a cylinder in* V.

3. If we assume (i) of the theorem, then the case of (ii) in which $\mathfrak{A}$ and $\mathfrak{B}$ are homologous to zero follows at once. For suppose $\mathfrak{A}$ and $\mathfrak{B}$ divide V into V', V'', and V''', with boundaries $\mathfrak{A}$, $\mathfrak{A}\cup\mathfrak{B}$, and $\mathfrak{B}$ respectively. Since $\mathfrak{A}$ is not null-homotopic we know that $g'\neq 0$. Construct a new surface, W, by attaching a disc, D, along $\mathfrak{B}$ to $V'\cup V''$. Now $\mathfrak{A}$ and $\mathfrak{B}$ are null-homotopic and therefore $\mathfrak{A}$ bounds a disc in W. Since $\mathfrak{A}$ is the boundary in W of V' and $V''\cup D$ and since V' is not a disc $V''\cup D$ must be. Thus $g''=g(V''\cup D)=0$, so V'' is a cylinder.

4. Part (i) of the theorem is an immediate consequence of Van Kampen's Theorem [3]. Let $\mathfrak{A}$ be a simple, closed, null-homotopic curve in V. The curve divides V into two components whose closures are V' and V'' with common boundary, $\mathfrak{A}$. Let T be an open tubular *nbhd* about a and let $V_1=V'\cup T$ and $V_2=V''\cup T$. $\{V_1, V_2\}$ is an open cover of V and $T=V_1\cap V_2$ is connected. Let $x_0\in\mathfrak{A}$; we compute $\pi_1(V, x_0)$ from $\pi_1(V_i, x_0)$ and $\pi_1(T, x_0)$. Let α, α_i be the homotopy class of $\mathfrak{A}$ in $\pi_1(T, x_0)$, $\pi_1(V_i, x_0)$, $i=1, 2$. $\pi_1(V_1, x_0)$ has generators $\alpha_1, A_i, B_i, i=1, \cdots, g'$, with one relation $\alpha_1=[A_1, B_1]\cdot[A_2, B_2]\cdot \cdots \cdot[A_{g'}, B_{g'}]=w_1$, where $[X, Y]$ is the commutator of X and Y and $g'=g(V')$. Similarly $\pi_1(V_2, x_0)$ has generators $\alpha_2, C_j, D_j, j=1, \cdots, g''$, with one relation $\alpha_2=[C_1, D_1]\cdot[C_2, D_2]\cdots[C_{g''}, D_{g''}]=w_2$, where $g''=g(V'')$. $\pi_1(T, x_0)$ has one generator, α, with no relations. Applying Van Kampen's Theorem we have $\pi_1(V, x_0)$ is generated by α, α_1, α_2, A_i, B_i, C_j, D_j; $i=1, \cdots, g'$; $j=1, \cdots, g''$ with the relations:

$$\alpha = \alpha_1 = \alpha_2 = w_1 = w_2.$$

Or $\pi_1(V, x_0)$ is generated by A_i, B_i, C_j, D_j with the one relation $w_1=w_2$. We show that the two assumptions g' and g'' nonzero and $\mathfrak{A}$ null-homotopic in V are incompatible. In $\pi_1(V, x_0)$ the class of $\mathfrak{A}$ is represented by the images of w_1 and w_2. Thus if this class is 1 in $\pi_1(V, x_0)$, we must be able to express w_1 as a word in the conjugates of powers of $w_1^{-1}w_2$.

LEMMA. *Let F be the free group with generators A_i, B_i, C_j, D_j; $i=1, \cdots, g'$ and $j=1, \cdots, g''$. Let $w_1=[A_1, B_1]\cdot \cdots \cdot[A_{g'}, B_{g'}]$ and $w_2=[C_1, D_1]\cdot \cdots \cdot[C_{g''}, D_{g''}]$. If N is the smallest, normal subgroup of F containing $w_1^{-1}w_2$ then $w_1\notin N$.*

PROOF. Let H be the subgroup of (3×3) real, triangular matrices generated by:

$$h_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad h_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Define two homomorphisms $\psi_i\colon F\to H$ as follows: $\psi_1(A_1)=\psi_2(C_1)=h_1$, $\psi_1(B_1)=\psi_2(D_1)=h_2$, and $\psi_2(A_1)=\psi_2(B_1)=\psi_1(C_1)=\psi_1(D_1)=I$; ψ_i of any other generator is I. Notice that $\psi_2(w_1)=I$, $\psi_1(w_1)=\psi_2(w_2)=h\neq I$, and that $[H, [H, H]]=I$. If $w_1\in N$, then w_1 is a word in the conjugates of the powers of $w_1^{-1}w_2$. Thus for some integer k, $\psi_1(w_1)=\psi_1(w_1)^{-k}$ and $\psi_2(w_1)=\psi_2(w_2)^k$. So we have the contradiction, $h=I$.

5. We now complete the proof of (ii). The curves $\mathfrak{A}$ and $\mathfrak{B}$ are freely homotopic, nonintersecting, simple closed curves in V which are not homologous to zero. Let V' and V'' be the closures of the components of $V-\mathfrak{A}\cup\mathfrak{B}$. V' and V'' have $\mathfrak{A}\cup\mathfrak{B}$ as their common boundary. Let M and N be disjoint tubular *nbhds* of $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Let $V_1=V'\cup M\cup N$ and $V_2=V''\cup M\cup N$; $V_1\cap V_2=M\cup N$. In order to compute the fundamental group of V we use Weinzweig's generalization of Van Kampen's Theorem [**4**].

Pick a point $x_0\in\mathfrak{A}$ and $y_0\in\mathfrak{B}$. Since both x_0 and y_0 are in $V_1\cap V_2$ we may pick arcs $d_i\colon I\to V_i$ such that $d_i(0)=x_0$ and $d_i(1)=y_0$. Let $\mathfrak{D}\colon I\to V$ be defined by:

$$\mathfrak{D}(t) = \begin{cases} d_1(2t), & \text{if } 0 \leqq t \leqq 1/2, \\ d_2(2-2t), & \text{if } 1/2 \leqq t \leqq 1. \end{cases}$$

Let α_i be the class of $\mathfrak{A}$ in $\pi_1(V_i, x_0)$, and β_i be the class of $d_i\mathfrak{B}d_i^{-1}$ in $\pi_1(V_i, x_0)$. Let α be the class of $\mathfrak{A}$ in $\pi_1(M, x_0)$ and β be the class of $\mathfrak{B}$ in $\pi_1(N, y_0)$. $\pi_1(M, x_0)$ and $\pi_1(N, y_0)$ are infinite cyclic with generators α and β, respectively. $\pi_1(V_1, x_0)$ has generators α_1, β_1, A_i, B_i; $i=1, \cdots, g'=g(V')$; with the one relation $\alpha_1=\beta_1[A_1, B_1]\cdot \cdots \cdot[A_{g'}, B_{g'}]=\beta_1 w_1$. $\pi_1(V_2, x_0)$ has generators $\alpha_2, \beta_2, C_j, D_j$; $j=1, \cdots, g''=g(V'')$; with the one relation $\alpha_2=\beta_2[C_1, D_1]\cdot \cdots \cdot [C_{g''}, D_{g''}]=\beta_2 w_2$. Using Theorem 2 of [**4**], we see that $\pi_1(V, x_0)$ is generated by: α, α_1, α_2, β, β_1, β_2, A_i, B_i, C_j, D_j, and δ, the class of $\mathfrak{D}$. The relations are: $\alpha=\alpha_1=\alpha_2$, $\beta=\beta_1=\delta\beta_2\delta^{-1}$, $\alpha_1=\beta_1\cdot w_1$, and $\alpha_2=\beta_2\cdot w_2$. Or $\pi_1(V, x_0)$ is generated by: β, δ, A_i, B_i, C_j, D_j with the relation $\beta w_2=\delta\beta\delta^{-1}w_1$.

Again we show that the assumptions that $\mathfrak{A}$ and $\mathfrak{B}$ are freely homotopic and that g' and g'' are nonzero are incompatible. If we let G be the quotient group obtained from $\pi_1(V, x_0)$ by introducing the relations $\beta=1$ and $\delta=1$, G is the group generated by A_i, B_i, C_j, D_j with the relation $w_1=w_2$. If $\mathfrak{A}$ and $\mathfrak{B}$ are freely homotopic then the image of w_1 in G is 1, but by the lemma of the preceding section this is possible only when either g' or g'' is zero.

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