ZEROES OF THE COHOMOLOGY OF THE STEENROD ALGEBRA

ARUNAS LIULEVICIUS[1]

1. **Introduction.** The paper is devoted to proving the analogue of the Adams' vanishing theorem [4] for the cohomology of the Steenrod algebra A over Z_p, where p is an odd prime. The result is used to obtain a better bound on the order of elements in the stable homotopy groups of spheres. The methods of proof are analogues of [4].

Let A_0 be the subalgebra of A consisting of 1 and Q_0 [9]; A_0 has a natural A-module structure consistent with the inclusion $A_0 \subset A$.

THEOREM 1. *Let M be any A_0-free A-module such that $M_t = 0$ for $t < m$. Then* $\mathrm{Ext}_A^{s,t}(M, Z_p) = 0$ *for* $t < m + (2p-1)s - 1$, $s \geqq 1$.

COROLLARY 1. $\mathrm{Ext}_A^{s,t}(Z_p, Z_p) = 0$ *for* $t < (2p-1)s - 2$, $s \geqq 1$.

THEOREM 2. *Let Π_r^S be the rth stable homotopy group of the sphere, p an odd prime. Then Π_r^S contains no p-elements of order* $> p^{[(r+2)/2(p-1)]}$.

2. **Preliminary computations.** Let A be the Steenrod algebra [9] over Z_p, p an odd prime. Let A_r be the subalgebra of A generated by 1 and Q_0, P^{p^k}, $k = 0, \cdots, r-1$ (we set $P^{-1} = 0$, $A_\infty = A$). Each A_r is a Hopf subalgebra of A_s, $s \geqq r$, therefore [10] A_s is free as a left (or right) A_r-module. The subalgebra A_0 is a left A_r-module, the module structure being consistent with the inclusion $A_0 \subset A_r$.

PROPOSITION 1. *If $s \geqq r$, then $A_s \otimes_{A_r} A_0$ is free as a left A_0-module.*

PROOF. Consider the graded dual A_s^* of A_s:

$$A_s^* = \Lambda_p[\tau_0, \cdots, \tau_s] \otimes_{Z_p} Z_p[\xi_1, \cdots, \xi_r]/I_r, \tag{1}$$

where I_r is the ideal in the polynomial ring generated by $\xi_1^{p^r}, \cdots, \xi_k^{p^{r-k+1}}, \cdots, \xi_r$ (see [9]). The proof is completed by exhibiting $(A_s \otimes_{A_r} A_0)^*$ as a subspace of $A_s^* \otimes A_0^*$, and proving that the former is a free left A_0^*-comodule. For this purpose it is convenient to replace τ_i and ξ_j in (1) by $c(\tau_i)$, $c(\xi_j)$, where c is the conjugation antiautomorphism.

We wish to study the groups $\mathrm{Ext}_A(Z_p, Z_p)$. Let us write $\beta \in (s, t)$ if $\beta \in \mathrm{Ext}_A^{s,t}(Z_p, Z_p)$. These groups have been computed completely

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in [5] for $t-s \leqq 2p(p-1)-1$. The results are as follows: there are classes

$$(2)\qquad \begin{array}{lll} 1 \in (0, 0), & \alpha_0 \in (1, 1), & h_i \in (1, 2p^i(p-1)), \\ \lambda_i \in (2, 2p^{i+1}(p-1)), & \rho_s \in (s, 2sp-s-1), & 2 \leqq s \leqq p, \end{array}$$

such that the following elements constitute Z_p-bases for $(s, *)$ in total degrees $t-s \leqq 2p(p-1)-1$:

$$(3)\qquad \begin{array}{ll} 1 \text{ for } (0, *) \alpha_0, h_0, h_1 \text{ for } (1, *) & \\ \alpha_0^s, \rho_s, \alpha_0^{s-2}\lambda_0, \alpha_0^{s-1}h_1 \text{ for } (s, *), & 2 \leqq s \leqq p-1, \\ \alpha_0^s \text{ for } (s, *) \quad \text{with } s > p. & \end{array}$$

The elements (2) also satisfy the relations

$$(4)\qquad \alpha_0 h_0 = 0, \qquad \alpha_0 \rho_s = 0, \qquad \alpha_0^{p-1}\lambda_0 = 0, \qquad \alpha_0^p h_1 = 0.$$

The information in (2)–(4) allows us to compute a good part of $\mathrm{Ext}_A(A_0, Z_p)$ (see [1]). In particular, we have

LEMMA 1. $\mathrm{Ext}_A^{s,t}(A_0, Z_p)=0$ *for* $1 \leqq s \leqq p$, $t<2ps-s-1$.

PROPOSITION 2. *If M is an A-module which is A_0-free and $M_t=0$ for $t<m$, then* $\mathrm{Ext}_A^{s,t}(M, Z_p)=0$ *for* $1 \leqq s \leqq p$, $t<m+2ps-s-1$.

PROOF. In the spirit of Lemma 3 of [3]. Induction and Five Lemma.

PROPOSITION 3. *Suppose that for any M as in Proposition 2 we have* $\mathrm{Ext}_A^{s,t}(M, Z_p)=0$ *for $t<m+F(s)$ for $s=1, \cdots, k$, then* $\mathrm{Ext}_A^{s+i,t}(M, Z_p)=0$ *for $t<m+F(k)+F(i)$, $i=1, \cdots, k$.*

PROOF. Consider a minimal resolution [2] of M as an A-module. Let N be the module of $(k-1)$-cycles. Then $N_t=0$ for $t<m+F(k)$. Since A and M are A_0-free, so is N. Applying the hypothesis of the proposition to N, we have $\mathrm{Ext}_A^{i,t}(N, Z_p)=0$ for $t<m+F(k)+F(i)$, $i=1, \cdots, k$. The proof is completed by remarking that

$$\mathrm{Ext}_A^{i,t}(N, Z_p) \cong \mathrm{Ext}_A^{k+i,t}(M, Z_p).$$

COROLLARY. *If M is any A_0-free A-module with $M_t=0$ for $t<m$, then* $\mathrm{Ext}_A^{s,t}(M, Z_p)=0$ *for $t<m+T(s)$, where $s>0$ and $T(rp+j)=r(2p^2-p-1)+2pj-j-1$, $j=1, \cdots, p$.*

PROPOSITION 4. *Let $i: A_r \to A$ be the inclusion map. Under the hypotheses of Proposition 3 (with $F(s) \geqq F(s-1)$)*

$$i^*: \mathrm{Ext}_A^{s,t}(M, Z_p) \to \mathrm{Ext}_{A_r}^{s,t}(M, Z_p)$$

is an isomorphism for $t < m + F(s-1) + 2p^{r+1}(p-1)$, *where* $s = 1, \cdots, k$.

PROOF. Consider

$$0 \to K \to A \otimes_{A_r} M \to M \to 0$$

since M and $A \otimes_{A_r} M$ are both A_0-free (Proposition 1), so is K. Also $K_t = 0$ for $t < m + 2p^{r+1}(p-1)$. Thus $\mathrm{Ext}_A^{s,t}(K, Z_p) = 0$ for $t < m + 2p^{r+1}(p-1) + F(s)$. Since $F(s) \geqq F(s-1)$ (a trivial assumption), the proposition follows from the remark that

$$\mathrm{Ext}_A^{s,t}(A \otimes_{A_r} M; Z_p) \cong \mathrm{Ext}_{A_r}^{s,t}(M, Z_p),$$

for A is free as a right A_r-module [10].

3. **The cohomology of A_1.** We wish to compute $\mathrm{Ext}_{A_1}(Z_p, Z_p)$; we shall use the method of [8]. Let $Q_1 = [P^1, Q_0]$ (see [9]); then we have the following relations:

$$Q_0Q_0 = 0, \quad Q_1Q_1 = 0, \quad [Q_0, Q_1] = 0, \quad (P^1)^p = 0, \quad [P^1, Q_1] = 0.$$

Let D be the exterior algebra generated by Q_1; D is a normal [10] Hopf subalgebra of A_1, and $G = A_1//D$ is a tensor product of an exterior algebra on $e = Q_0 + A_1\overline{D}$, and a truncated polynomial algebra on $a = P^1 + A_1\overline{D}$. It is well known that

$$\mathrm{Ext}_G^{*,*}(Z_p, Z_p) = Z_p[\alpha] \otimes \Lambda_p[\mu] \otimes Z_p[\lambda],$$

where $\alpha \in (1, 1)$, $\mu \in (1, 2p-2)$, $\lambda \in (2, 2p(p-1))$; similarly,

$$\mathrm{Ext}_D^{*,*}(Z_p, Z_p) = Z_p[\beta],$$

where $\beta \in (1, 2p-1)$.

We can describe minimal resolutions for Z_p over G and D very easily: for the minimal resolution $Y = G \otimes \overline{Y}$ we take a complex with G-free generators $[\alpha^k \mu^\epsilon \lambda^m]$, where $\epsilon = 0, 1, k, m = 0, 1, 2, \cdots$, and define the differential d' as follows:

$$d'[\alpha^k \mu \lambda^m] = e[\alpha^{k-1}\mu\lambda^m] + a[\alpha^k \lambda^m],$$
$$d'[\alpha^k \lambda^{m+1}] = e[\alpha^{k-1}\lambda^{m+1}] + a^{p-1}[\alpha^k \mu \lambda^m].$$

Similarly, for the minimal resolution $W = D \otimes \overline{W}$ we take a complex with D-free generators $[\beta^r]$, $r = 0, 1, 2, \cdots$, and differential d'':

$$d''[\beta^r] = Q_1[\beta^{r-1}].$$

REMARK. The reader is cautioned that we are using the (reasonable) sign convention: any time two maps (with degree and grading) are switched past each other, we multiply by -1 raised to the product of total degrees. Example: a D-map f of degree 1 and grading 0 satisfies $f(Q_1 m) = -Q_1 f(m)$.

We construct an A_1-resolution of Z_p by introducing a suitable differential d in $A_1 \otimes \overline{Y} \otimes \overline{W}$ (compare [8]). We let $d = \sum_{k=0}^{\infty} d_k$, where d_0 is induced by d'':

$$d_0[\alpha^k \mu^\epsilon \lambda^m] \otimes [\beta^r] = (-1)^\epsilon [\alpha^k \mu \lambda^m] \otimes Q_1[\beta^{r-1}] = Q_1[\alpha^k \mu^\epsilon \lambda^m] \otimes [\beta^{r-1}],$$

and d_k, $k \geqq 1$, are defined as follows:

$$\begin{aligned}
d_1([\alpha^k \mu \lambda^m] \otimes [\beta^r]) &= e[\alpha^{k-1} \mu \lambda^m] \otimes [\beta^r] + a[\alpha^k \lambda^m] \otimes [\beta^r], \\
d_1([\alpha^k \lambda^{m+1}] \otimes [\beta^r]) &= e[\alpha^k \lambda^{m+1}] \otimes [\beta^r] + a^{p-1}[\alpha^k \mu \lambda^m] \otimes [\beta^r], \\
d_2([\alpha^k \mu \lambda^m] \otimes [\beta^r]) &= -(r+1)[\alpha^{k-1} \lambda^m] \otimes [\beta^{r+1}], \\
d_q([\alpha^k \mu \lambda^m] \otimes [\beta^r]) &= 0, \qquad q \geqq 3, \\
d_j([\alpha^k \lambda^{m+1}] \otimes [\beta^r]) &= (j-1)! \binom{r+j-1}{j-1} a^{p-j}[\alpha^{k-j+1} \mu \lambda^m] \otimes [\beta^{r+j-1}], \\
&\hspace{18em} \text{for } j \geqq 2.
\end{aligned}$$

Since d_k, $k = 0, 1, \cdots$, satisfy the conditions of Theorem 1 of [8], $A_1 \otimes \overline{Y} \otimes \overline{W}$ yields an A_1-resolution of Z_p.

The elements $[\alpha]$, $[\lambda]$, $[1] \otimes [\beta^p]$, $[\mu] \otimes [\beta^j]$, $j = 0, 1, \cdots, p-2$ yield elements in $\mathrm{Tor}^{A_1}_{*,*}(Z_p, Z_p)$. Denote by α, λ, ω, ρ_{j+1}, $j = 0, 1, \cdots, p-2$, their duals in $\mathrm{Ext}^{*,*}_{A_1}(Z_p, Z_p)$. We then immediately have:

PROPOSITION 5. $\mathrm{Ext}_{A_1}(Z_p, Z_p)$ *is a free* $Z_p[\omega]$*-module with a set of free generators given by the elements*

$$\alpha^k, \qquad \alpha^j \lambda^m, \qquad \rho_s \lambda^m,$$

where $k = 0, 1, 2, \cdots$, $j = 0, 1, \cdots, p-2$, $s = 1, \cdots, p-1$. *The elements satisfy the relations*

$$\alpha \rho_s = 0, \qquad \alpha^{p-1} \lambda = 0.$$

The next proposition is now trivial.

PROPOSITION 6. (i) $\mathrm{Ext}^{s,t}_{A_1}(A_0, Z_p) = 0$ *if* $t < (2p-1)s - 1$, $s > 0$;

(ii) *multiplication by* ω *is an isomorphism in* $\mathrm{Ext}^{s,t}_{A_1}(A_0, Z_p)$ *for* $t < (2p-1)s + 2p^2 - 6p + 1$.

COROLLARY. *If* M *is an* A_1*-module which is* A_0*-free, and* $M_t = 0$ *for* $t < m$, *then* $\mathrm{Ext}^{s,t}_{A_1}(M, Z_p) = 0$ *if* $t < m + (2p-1)s - 1$ *and* $s > 0$.

REMARK. We cannot prove Proposition 6 (ii) for general A_0-free M. However, it seems to be true for $M = \overline{A}/A\overline{A}_0$: that is, $\mathrm{Ext}^{s,t}_A(Z_p, Z_p)$ seems to be periodic in a small neighborhood of the line $t = (2p-1)s - 2$.

4. **Proof of Theorems 1 and 2.** Write $s=rp+i$, $i=1, \cdots, p$. Theorem 1 is proved by induction on r. For $r=0$ this is Proposition 2. We suppose that the theorem has been proved for $r'<r$ and all A_0-free M; we then estimate the zeroes in dimensions $rp+i$ by using Proposition 3. Here Proposition 4 gives an isomorphism with $\mathrm{Ext}_{A_1}^{s,t}(M, Z_p)$ in a neighborhood of the line $t=m+(2p-1)s-2$. According to the corollary of Proposition 6, this enables us to prove that $\mathrm{Ext}_A^{s,t}(M, Z_p)=0$ for $t<m+(2p-1)s-1$ for $s=rp+i$, $i=1, \cdots, p$, which completes the inductive step.

Corollary 1 follows from the observation that $N=\bar{A}/A\bar{A}_0$ is A_0-free and $N_t=0$ for $t<2p-2$.

Proposition 6(ii) and Proposition 4 with $F(s)=(2p-1)s-1$ prove the following:

COROLLARY 2. $\mathrm{Ext}_A^{s,t}(A_0, Z_p)\cong\mathrm{Ext}_A^{s+p,t+2p^2-p}(A_0, Z_p)$ *for* $t<(2p-1)s+2p^2-6p+1$.

Theorem 2 is an immediate consequence of Corollary 1 and the Adams spectral sequence [1].

REMARK. Theorem 2 shows that there are no elements of order $>p^{p^k}$ in dimension $r=2p^k(p-1)-1$. Since the mod p Hopf invariant is trivial for $k>0$ [7], there are no elements of order $>p^{p^k-1}$ in these dimensions. Theorem 2 should be compared with Theorem 7 of [6].

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