

ZEREOES OF THE COHOMOLOGY OF THE STEENROD ALGEBRA

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1. Introduction. The paper is devoted to proving the analogue of the Adams' vanishing theorem [4] for the cohomology of the Steenrod algebra A over Z_p , where p is an odd prime. The result is used to obtain a better bound on the order of elements in the stable homotopy groups of spheres. The methods of proof are analogues of [4].

Let A_0 be the subalgebra of A consisting of 1 and Q_0 [9]; A_0 has a natural A -module structure consistent with the inclusion $A_0 \subset A$.

THEOREM 1. *Let M be any A_0 -free A -module such that $M_t = 0$ for $t < m$. Then $\text{Ext}_A^{s,t}(M, Z_p) = 0$ for $t < m + (2p-1)s - 1$, $s \geq 1$.*

COROLLARY 1. $\text{Ext}_A^{s,t}(Z_p, Z_p) = 0$ for $t < (2p-1)s - 2$, $s \geq 1$.

THEOREM 2. *Let Π_r^s be the r th stable homotopy group of the sphere, p an odd prime. Then Π_r^s contains no p -elements of order $> p^{\lfloor (r+2)/2(p-1) \rfloor}$.*

2. Preliminary computations. Let A be the Steenrod algebra [9] over Z_p , p an odd prime. Let A_r be the subalgebra of A generated by 1 and $Q_0, P^{p^k}, k=0, \dots, r-1$ (we set $P^{-1}=0, A_\infty=A$). Each A_r is a Hopf subalgebra of $A_s, s \geq r$, therefore [10] A_s is free as a left (or right) A_r -module. The subalgebra A_0 is a left A_r -module, the module structure being consistent with the inclusion $A_0 \subset A_r$.

PROPOSITION 1. *If $s \geq r$, then $A_s \otimes_{A_r} A_0$ is free as a left A_0 -module.*

PROOF. Consider the graded dual A_s^* of A_s :

$$(1) \quad A_s^* = \Delta_p[\tau_0, \dots, \tau_s] \otimes_{Z_p} Z_p[\xi_1, \dots, \xi_r] / I_r,$$

where I_r is the ideal in the polynomial ring generated by $\xi_1^p, \dots, \xi_k^{p^{k+1}}, \dots, \xi_r$ (see [9]). The proof is completed by exhibiting $(A_s \otimes_{A_r} A_0)^*$ as a subspace of $A_s^* \otimes A_0^*$, and proving that the former is a free left A_0^* -comodule. For this purpose it is convenient to replace τ_i and ξ_j in (1) by $c(\tau_i), c(\xi_j)$, where c is the conjugation antiautomorphism.

We wish to study the groups $\text{Ext}_A(Z_p, Z_p)$. Let us write $\beta \in (s, t)$ if $\beta \in \text{Ext}_A^{s,t}(Z_p, Z_p)$. These groups have been computed completely

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in [5] for $t-s \leq 2p(p-1) - 1$. The results are as follows: there are classes

$$(2) \quad \begin{aligned} &1 \in (0, 0), \quad \alpha_0 \in (1, 1), \quad h_i \in (1, 2p^i(p-1)), \\ &\lambda_i \in (2, 2p^{i+1}(p-1)), \quad \rho_s \in (s, 2sp - s - 1), \quad 2 \leq s \leq p, \end{aligned}$$

such that the following elements constitute Z_p -bases for $(s, *)$ in total degrees $t-s \leq 2p(p-1) - 1$:

$$(3) \quad \begin{aligned} &1 \text{ for } (0, *)\alpha_0, h_0, h_1 \text{ for } (1, *) \\ &\alpha_0^s, \rho_s, \alpha_0^{s-2}\lambda_0, \alpha_0^{s-1}h_1 \text{ for } (s, *), \quad 2 \leq s \leq p-1, \\ &\alpha_0^s \text{ for } (s, *) \quad \text{with } s > p. \end{aligned}$$

The elements (2) also satisfy the relations

$$(4) \quad \alpha_0 h_0 = 0, \quad \alpha_0 \rho_s = 0, \quad \alpha_0^{p-1} \lambda_0 = 0, \quad \alpha_0^p h_1 = 0.$$

The information in (2)–(4) allows us to compute a good part of $\text{Ext}_A(A_0, Z_p)$ (see [1]). In particular, we have

LEMMA 1. $\text{Ext}_A^{s,t}(A_0, Z_p) = 0$ for $1 \leq s \leq p, t < 2ps - s - 1$.

PROPOSITION 2. *If M is an A -module which is A_0 -free and $M_t = 0$ for $t < m$, then $\text{Ext}_A^{s,t}(M, Z_p) = 0$ for $1 \leq s \leq p, t < m + 2ps - s - 1$.*

PROOF. In the spirit of Lemma 3 of [3]. Induction and Five Lemma.

PROPOSITION 3. *Suppose that for any M as in Proposition 2 we have $\text{Ext}_A^{s,t}(M, Z_p) = 0$ for $t < m + F(s)$ for $s = 1, \dots, k$, then $\text{Ext}_A^{s+t,t}(M, Z_p) = 0$ for $t < m + F(k) + F(i), i = 1, \dots, k$.*

PROOF. Consider a minimal resolution [2] of M as an A -module. Let N be the module of $(k-1)$ -cycles. Then $N_t = 0$ for $t < m + F(k)$. Since A and M are A_0 -free, so is N . Applying the hypothesis of the proposition to N , we have $\text{Ext}_A^{t,t}(N, Z_p) = 0$ for $t < m + F(k) + F(i), i = 1, \dots, k$. The proof is completed by remarking that

$$\text{Ext}_A^{i,t}(N, Z_p) \cong \text{Ext}_A^{k+i,t}(M, Z_p).$$

COROLLARY. *If M is any A_0 -free A -module with $M_t = 0$ for $t < m$, then $\text{Ext}_A^{s,t}(M, Z_p) = 0$ for $t < m + T(s)$, where $s > 0$ and $T(rp+j) = r(2p^2 - p - 1) + 2pj - j - 1, j = 1, \dots, p$.*

PROPOSITION 4. *Let $i: A_r \rightarrow A$ be the inclusion map. Under the hypotheses of Proposition 3 (with $F(s) \geq F(s-1)$)*

$$i^* : \text{Ext}_A^{s,t}(M, Z_p) \rightarrow \text{Ext}_{A_r}^{s,t}(M, Z_p)$$

is an isomorphism for $t < m + F(s - 1) + 2p^{r+1}(p - 1)$, where $s = 1, \dots, k$.

PROOF. Consider

$$0 \rightarrow K \rightarrow A \otimes_{A_r} M \rightarrow M \rightarrow 0$$

since M and $A \otimes_{A_r} M$ are both A_0 -free (Proposition 1), so is K . Also $K_t = 0$ for $t < m + 2p^{r+1}(p - 1)$. Thus $\text{Ext}_A^{s,t}(K, Z_p) = 0$ for $t < m + 2p^{r+1}(p - 1) + F(s)$. Since $F(s) \geq F(s - 1)$ (a trivial assumption), the proposition follows from the remark that

$$\text{Ext}_A^{s,t}(A \otimes_{A_r} M; Z_p) \cong \text{Ext}_{A_r}^{s,t}(M, Z_p),$$

for A is free as a right A_r -module [10].

3. **The cohomology of A_1 .** We wish to compute $\text{Ext}_{A_1}(Z_p, Z_p)$; we shall use the method of [8]. Let $Q_1 = [P^1, Q_0]$ (see [9]); then we have the following relations:

$$Q_0 Q_0 = 0, \quad Q_1 Q_1 = 0, \quad [Q_0, Q_1] = 0, \quad (P^1)^p = 0, \quad [P^1, Q_1] = 0.$$

Let D be the exterior algebra generated by Q_1 ; D is a normal [10] Hopf subalgebra of A_1 , and $G = A_1/D$ is a tensor product of an exterior algebra on $e = Q_0 + A_1 \bar{D}$, and a truncated polynomial algebra on $a = P^1 + A_1 \bar{D}$. It is well known that

$$\text{Ext}_G^{*,*}(Z_p, Z_p) = Z_p[\alpha] \otimes \Lambda_p[\mu] \otimes Z_p[\lambda],$$

where $\alpha \in (1, 1)$, $\mu \in (1, 2p - 2)$, $\lambda \in (2, 2p(p - 1))$; similarly,

$$\text{Ext}_D^{*,*}(Z_p, Z_p) = Z_p[\beta],$$

where $\beta \in (1, 2p - 1)$.

We can describe minimal resolutions for Z_p over G and D very easily: for the minimal resolution $Y = G \otimes \bar{Y}$ we take a complex with G -free generators $[\alpha^k \mu^\epsilon \lambda^m]$, where $\epsilon = 0, 1$, $k, m = 0, 1, 2, \dots$, and define the differential d' as follows:

$$\begin{aligned} d'[\alpha^k \mu \lambda^m] &= e[\alpha^{k-1} \mu \lambda^m] + a[\alpha^k \lambda^m], \\ d'[\alpha^k \lambda^{m+1}] &= e[\alpha^{k-1} \lambda^{m+1}] + a^{p-1}[\alpha^k \mu \lambda^m]. \end{aligned}$$

Similarly, for the minimal resolution $W = D \otimes \bar{W}$ we take a complex with D -free generators $[\beta^r]$, $r = 0, 1, 2, \dots$, and differential d'' :

$$d''[\beta^r] = Q_1[\beta^{r-1}].$$

REMARK. The reader is cautioned that we are using the (reasonable) sign convention: any time two maps (with degree and grading)

are switched past each other, we multiply by -1 raised to the product of total degrees. Example: a D -map f of degree 1 and grading 0 satisfies $f(Q_1 m) = -Q_1 f(m)$.

We construct an A_1 -resolution of Z_p by introducing a suitable differential d in $A_1 \otimes \overline{Y} \otimes \overline{W}$ (compare [8]). We let $d = \sum_{k=0}^{\infty} d_k$, where d_0 is induced by d'' :

$$d_0[\alpha^k \mu^e \lambda^m] \otimes [\beta^r] = (-1)^e [\alpha^k \mu \lambda^m] \otimes Q_1[\beta^{r-1}] = Q_1[\alpha^k \mu^e \lambda^m] \otimes [\beta^{r-1}],$$

and $d_k, k \geq 1$, are defined as follows:

$$\begin{aligned} d_1([\alpha^k \mu \lambda^m] \otimes [\beta^r]) &= e[\alpha^{k-1} \mu \lambda^m] \otimes [\beta^r] + a[\alpha^k \lambda^m] \otimes [\beta^r], \\ d_1([\alpha^k \lambda^{m+1}] \otimes [\beta^r]) &= e[\alpha^k \lambda^{m+1}] \otimes [\beta^r] + a^{p-1}[\alpha^k \mu \lambda^m] \otimes [\beta^r], \\ d_2([\alpha^k \mu \lambda^m] \otimes [\beta^r]) &= -(r+1)[\alpha^{k-1} \lambda^m] \otimes [\beta^{r+1}], \\ d_q([\alpha^k \mu \lambda^m] \otimes [\beta^r]) &= 0, \quad q \geq 3, \\ d_j([\alpha^k \lambda^{m+1}] \otimes [\beta^r]) &= (j-1)! \binom{r+j-1}{j-1} a^{p-j} [\alpha^{k-j+1} \mu \lambda^m] \otimes [\beta^{r+j-1}], \end{aligned}$$

for $j \geq 2$.

Since $d_k, k=0, 1, \dots$, satisfy the conditions of Theorem 1 of [8], $A_1 \otimes \overline{Y} \otimes \overline{W}$ yields an A_1 -resolution of Z_p .

The elements $[\alpha], [\lambda], [1] \otimes [\beta^p], [\mu] \otimes [\beta^i], j=0, 1, \dots, p-2$ yield elements in $\text{Tor}_{*,*}^{A_1}(Z_p, Z_p)$. Denote by $\alpha, \lambda, \omega, \rho_{j+1}, j=0, 1, \dots, p-2$, their duals in $\text{Ext}_{A_1}^{*,*}(Z_p, Z_p)$. We then immediately have:

PROPOSITION 5. $\text{Ext}_{A_1}(Z_p, Z_p)$ is a free $Z_p[\omega]$ -module with a set of free generators given by the elements

$$\alpha^k, \quad \alpha^j \lambda^m, \quad \rho_s \lambda^m,$$

where $k=0, 1, 2, \dots, j=0, 1, \dots, p-2, s=1, \dots, p-1$. The elements satisfy the relations

$$\alpha \rho_s = 0, \quad \alpha^{p-1} \lambda = 0.$$

The next proposition is now trivial.

PROPOSITION 6. (i) $\text{Ext}_{A_1}^{s,t}(A_0, Z_p) = 0$ if $t < (2p-1)s-1, s > 0$;
 (ii) multiplication by ω is an isomorphism in $\text{Ext}_{A_1}^{s,t}(A_0, Z_p)$ for $t < (2p-1)s+2p^2-6p+1$.

COROLLARY. If M is an A_1 -module which is A_0 -free, and $M_t = 0$ for $t < m$, then $\text{Ext}_{A_1}^{s,t}(M, Z_p) = 0$ if $t < m + (2p-1)s-1$ and $s > 0$.

REMARK. We cannot prove Proposition 6 (ii) for general A_0 -free M . However, it seems to be true for $M = \overline{A}/A\overline{A}_0$: that is, $\text{Ext}_{A_1}^{s,t}(Z_p, Z_p)$ seems to be periodic in a small neighborhood of the line $t = (2p-1)s - 2$.

4. **Proof of Theorems 1 and 2.** Write $s = rp + i$, $i = 1, \dots, p$. Theorem 1 is proved by induction on r . For $r = 0$ this is Proposition 2. We suppose that the theorem has been proved for $r' < r$ and all A_0 -free M ; we then estimate the zeroes in dimensions $rp + i$ by using Proposition 3. Here Proposition 4 gives an isomorphism with $\text{Ext}_{A_1}^{s,t}(M, Z_p)$ in a neighborhood of the line $t = m + (2p - 1)s - 2$. According to the corollary of Proposition 6, this enables us to prove that $\text{Ext}_{A_1}^{s,t}(M, Z_p) = 0$ for $t < m + (2p - 1)s - 1$ for $s = rp + i$, $i = 1, \dots, p$, which completes the inductive step.

Corollary 1 follows from the observation that $N = \bar{A}/A\bar{A}_0$ is A_0 -free and $N_t = 0$ for $t < 2p - 2$.

Proposition 6(ii) and Proposition 4 with $F(s) = (2p - 1)s - 1$ prove the following:

COROLLARY 2. $\text{Ext}_A^{s,t}(A_0, Z_p) \cong \text{Ext}_A^{s+p, t+2p^2-p}(A_0, Z_p)$ for $t < (2p - 1)s + 2p^2 - 6p + 1$.

Theorem 2 is an immediate consequence of Corollary 1 and the Adams spectral sequence [1].

REMARK. Theorem 2 shows that there are no elements of order $> p^k$ in dimension $r = 2p^k(p - 1) - 1$. Since the mod p Hopf invariant is trivial for $k > 0$ [7], there are no elements of order $> p^{p^k - 1}$ in these dimensions. Theorem 2 should be compared with Theorem 7 of [6].

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