ON THE MAXIMALITY THEOREM OF WERMER

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In this note we give a new proof of the following maximality theorem of J. Wermer [**5**]:

THEOREM. *Let C denote the sup-norm algebra of all continuous functions on the unit circle, and let A denote the subalgebra of those functions which have analytic extensions to the interior. Then A is a maximal closed subalgebra of C.*

As our proof depends only on the invariant subspace theorem for H^1 [**2**, Theorem 7]; see also [**4**], where H^1 denotes the L^1 closure of A, and the F. and M. Riesz theorem [**3**, p. 47], it works in any situation in which these two theorems are valid. Some algebras in which both theorems are valid were considered by Bishop [**1**]. Even though the proof extracted by Cohen [**3**, p. 94] from the proof of Wermer's theorem by Hoffman and Singer is at most equally short and certainly more elementary (as it does not use the F. and M. Riesz theorem), our proof seems to be of independent interest because of its generality, and because of the connection it exhibits between the maximality of an algebra and the properties of its annihilating measures.

PROOF OF WERMER'S THEOREM. Let B be a proper closed subalgebra of C containing A. We have to show that $B \subseteq A$, or equivalently

$$B^{\perp} \supseteq A^{\perp}, \tag{1}$$

where $\perp$ denotes the set of all orthogonal measures. Clearly $B^{\perp} \neq \{0\}$. Let $H_0^1 = \{f \in H^1 : \int f\, dm = 0\}$, where $dm = d\theta/2\pi$ denotes the normalized Lebesgue measure of the circle. By F. and M. Riesz theorem, $B^{\perp} \subseteq H_0^1\, dm$, and, since B is an algebra,

$$gB^{\perp} \subseteq B^{\perp} \tag{2}$$

for every $g \in B$. Hence, identifying fdm with f for $fdm \in B^{\perp}$, $B^{\perp}$ is a weak* closed, and therefore norm-closed, subspace of H^1, which by (2) is invariant under multiplication by $z = e^{i\theta}$. Hence by the invariant subspace theorem for H^1,

$$B^{\perp} = qH_0^1 \tag{3}$$

for some inner function q. Let $\mu \in A^{\perp}$. Then $\mu = fdm$ for some $f \in H_0^1$, by F. and M. Riesz theorem. By (2) and (3), $gqH_0^1 \subseteq qH_0^1$ for

Received by the editors August 3, 1963.

[1] Both authors were partially supported by NSF Grant G-18974 of the University of California, Berkeley. The second author was also partially supported by U.S. Air Force Grant AFOSR 407-63 of Yale University.

g in B, so $gH_0^1 \subseteq H_0^1$; in particular, $gf \in H_0^1$, therefore $g \perp f dm$. Thus $\mu \in B^\perp$, which proves (1).

References

1. E. Bishop, *A general Rudin-Carleson theorem*, Proc. Amer. Math. Soc. **13** (1962), 140–143.

2. K. de Leeuw and W. Rudin, *Extreme points and extreme problems in H_1*, Pacific J. Math. **8** (1958), 467–485.

3. K. Hoffman, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.

4. T. P. Srinivasan, *Simply invariant subspaces and generalized analytic functions*, (to appear).

5. J. Wermer, *On algebras of continuous functions*, Proc. Amer. Math. Soc. **4** (1953), 866–869.

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