

SOME BOUNDARY VALUE PROBLEMS FOR LINEAR DIFFERENTIAL SYSTEMS¹

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1. **Introduction.** Let $A(t)$ and $f(t)$ be $n \times n$ and $n \times 1$ matrices, respectively, continuous on an interval $[a, b]$. In [1], J. B. Garner and L. P. Burton consider the boundary value problems

$$(1) \quad y' = Ay + f, \quad y_i(a) = \beta_i \quad (1 \leq i < n), \quad y_n(b) = \beta_n$$

and

$$(2) \quad y' = Ay + f, \quad y_1(a) = \beta_1, \quad y_i(c) = \beta_i \quad (1 < i < n, a < c < b), \\ y_n(b) = \beta_n,$$

and prove:

THEOREM A. *If, for each i and j ($1 \leq i \leq n, 1 \leq j \leq n, i \neq j$), $a_{ij}a_{in}a_{jn} > 0$ and $a_{in}a_{ni} > 0$ on $[a, b]$, the problem (1) has a unique solution;*

THEOREM B. *Under certain conditions on $A(t)$, too lengthy to give here, the problem (2) has a unique solution.*

The authors note that Theorem A has a dual in which the roles of a and b are interchanged, provided that $a_{ij}a_{in}a_{jn} < 0$ is assumed.

The purpose here is to obtain theorems corresponding to Theorem A and its dual, with considerably less restriction on $A(t)$, and to use these results to obtain as a direct consequence a theorem corresponding to Theorem B.

2. **The two-point problem.** As usual, we rephrase the problem in terms of the homogeneous system. Let N be fixed ($1 \leq N \leq n$); let $Q = (\delta_{iN}\delta_{jN})$, and let $P = E - Q$ (E being the $n \times n$ identity); let $\beta = \text{col}(\beta_i)$. Let $z(t)$ be a solution of $y' = Ay + f$ which does not satisfy

$$(3) \quad y' = Ay + f, \quad Py(a) + Qy(b) = \beta.$$

If X is any nonsingular solution of $X' = AX$, the general solution of $y' = Ay + f$ can be written in the form $Xc + z$, and our boundary condition reduces to

$$[PX(a) + QX(b)] \cdot c = \beta - Pz(a) - Qz(b) \neq 0.$$

Thus (3) has a unique solution if and only if the equation $X' = AX$

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has a nonsingular solution for which $PX(a) + QX(b)$ is nonsingular. We may assume that $X(a) = E$; our condition is then that $x_N(b) \neq 0$ if $x' = Ax$ and $x_i(a) = \delta_{iN}$ ($1 \leq i \leq n$).

For convenience we list the following conditions and definitions.

(0) $a_{ii}(t) \equiv 0$ ($1 \leq i \leq n$).

(I) For N fixed ($1 \leq N \leq n$), there exist $K \neq N$ ($1 \leq K \leq n$) and m_K ($1 \leq m_K \leq n - 1$) such that no product $a_{Kj_1}(t_0)a_{j_1j_2}(t_1) \cdots a_{j_mN}(t_m)$, with at most $m_K + 1$ factors, changes sign on $a \leq t_i \leq b$ ($0 \leq i \leq m$), each such product has the same sign, and one such product, with at most m_K factors, is nonzero at $t = a$. Let s_{KN} be 1 or -1 , according as this last product is positive or negative at $t = a$; let $s_{NN} = 1$.

(II) For N fixed, (I) holds for each $K \neq N$; $s_{KN}a_{NK}(t) \geq 0$; the m_K 's may be taken equal.

(III) $x' = Ax$ and $x_i(a) = \delta_{iN}$ ($1 \leq i \leq n$).

LEMMA 1. If $x' = Ax$ and (0) holds, and if, for a fixed j_i , σ_i is the set of integers including $1, \dots, n$, but excluding j_i , then

$$(4i) \quad x_{j_0}(t_0) = x_{j_0}(a) + \sum_{j_1 \in \sigma_0} \int_a^{t_0} a_{j_0j_1}(t_1)x_{j_1}(t_1)dt_1;$$

$$(4ii) \quad x_{j_0}(t_0) = x_{j_0}(a) + \sum_{h=1}^m \sum_{j_1 \in \sigma_0} \cdots \sum_{j_h \in \sigma_{h-1}} \int_a^{t_0} dt_1 \cdots \\ \cdot \int_a^{t_{h-2}} dt_{h-1} \int_a^{t_{h-1}} x_{j_h}(a) \prod_{i=0}^{h-1} a_{j_i j_{i+1}}(t_{i+1}) dt_h \\ + \sum_{j_1 \in \sigma_0} \cdots \sum_{j_{m+1} \in \sigma_m} \int_a^{t_0} dt_1 \cdots \\ \cdot \int_a^{t_{m-1}} dt_m \int_a^{t_m} x_{j_{m+1}}(t_{m+1}) \prod_{i=0}^m a_{j_i j_{i+1}}(t_{i+1}) dt_{m+1}, \quad m \geq 1.$$

PROOF. Integration of x'_{j_0} gives (4i) and, in fact, a similar expression for each x_{j_i} . Substituting these expressions into the right-hand side of (4i), and continuing the process, gives (4ii).

LEMMA 2. If (0), (I), and (III) hold, there is a $\delta > 0$ such that $s_{KN}x_K(t) > 0$ on $(a, a + \delta)$.

PROOF. In (4), let $j_0 = K$ and $j_{m+1} = N$. By (III), the first term in each of (4i) and (4ii) is zero. Since all possible products with $m + 1$ factors and of the type in (I) occur in the last set of terms in (4i) or (4ii), proper choice of m will cause to appear a product involving x_N which is nonzero at a . This term, and indeed all terms involving x_N ,

have the sign of s_{KN} . All other terms in the last set are zero at a ; all nonzero terms in the second set of terms in (4ii) involve $x_N(a)$, and so have the sign of s_{KN} . Hence, for small positive δ , $s_{KN}x_E(t) > 0$ on $(a, a + \delta)$.

COROLLARY 1. *If (0), (II) and (III) hold, there is a $\delta > 0$ such that*

$$(5) \quad s_{iN}x_i(t) > 0 \quad (1 \leq i \leq n)$$

on $(a, a + \delta)$.

LEMMA 3. *If (0), (II) and (III) hold, then (5) holds on $(a, b]$.*

PROOF. By Corollary 1, there is a δ such that $0 < \delta < b - a$ and for which (5) holds on $(a, a + \delta)$. Let $a < c < a + \delta$, and let $P(t) = x_1(t) \cdots x_n(t)$. Then

$$P(t) = P(c) \exp \int_c^t \sum_{k, j=1; k \neq j}^n a_{kj}x_j/x_k dt$$

on $(c, a + \delta)$. Now, for each k and j such that $k \neq j$, we have $a_{kj}x_j/x_k \geq 0$. Indeed, if $k = N$ or $j = N$ this follows from the condition $s_{iN}a_{Ni}(t) \geq 0$ in (II) and from the conclusion of Lemma 2. If $j \neq N$ and $k \neq N$, let m be the common value of the m_i 's in (II). There is a product P_{jN} of the form in (I) (with $K = j$), with at most m factors, such that $s_{jN}P_{jN}(a) > 0$. Since $a_{kj}P_{jN}$ is a product like those in (I), with at most $m + 1$ factors, we have $s_{kN}a_{kj}P_{jN} \geq 0$. Hence we can write $a_{kj}x_j/x_k = [a_{kj}P_{jN}(a)/x_k] \cdot [x_j/P_{jN}(a)]$, with each factor non-negative. Thus the statement is verified. From this, $|P(t)| \geq |P(c)|$ on $(c, a + \delta)$. Since the inequality must hold even for $t = a + \delta$, then $P(a + \delta) \neq 0$.

Now let Δ be the lub of the set of δ 's such that $0 < \delta < b - a$ and for which (5) holds on $(a, a + \delta)$. Clearly (5) holds on $(a, a + \Delta)$, and so (by the above argument) (5) holds at $t = a + \Delta$. Unless $\Delta = b - a$, the continuity of $P(t)$ gives a contradiction to the lub property of Δ . This completes the proof.

The following theorem, corresponding to Theorem A, is now almost immediate.

THEOREM 1. *If (II) holds for products excluding the $a_{ii}(t)$'s, the problem (3) has a unique solution.*

PROOF. The proof depends only on Lemma 3, and so we must show that Lemma 3 holds even without condition (0). To this end, let $z(t) = G(t)x(t)$, where $x(t)$ satisfies (III) and $G(t)$ is the diagonal matrix for which $g_{ii}(t) = \exp \int_a^t -a_{ii}(s) ds$. Then $z' = B(t)z$, where $b_{ii}(t) \equiv 0$ and, for $i \neq j$, $b_{ij}(t) = a_{ij}(t) \exp \int_a^t [a_{jj}(s) - a_{ii}(s)] ds$. Thus if (II) holds for

$A(t)$ then (II) holds for $B(t)$, with the same s_{iN} 's; if (III) holds for $x(t)$ then (III) holds for $z(t)$; and (0) holds for $B(t)$. Thus Lemma 3 as it stands applies to $z(t)$, and also, since $x_i(t) = z_i(t) \exp \int_a^t a_{ii}(s) ds$, to $x(t)$. Thus condition (0) can be eliminated from Lemma 3, and the proof is complete.

The theorem corresponding to the dual of Theorem A is contained in the following statements. Corresponding to (I), (II), and (III) we have:

(I') As (I), except that the products with an odd number of factors and those with an even number of factors differ in sign, and there is at least one product, say with r factors ($r \leq m_K$), which is nonzero at b . Let $(-1)^r s'_{KN}$ be 1 or -1 according as this last product is positive or negative at b . Let $s'_{NN} = 1$.

(II') For some fixed N , (I) holds for each $K \neq N$ ($1 \leq K \leq n$); $a_{NK} s'_{KN} \leq 0$; and the m_K 's may all be taken equal.

(III') $x' = Ax$, and $x_i(b) = \delta_{iN}$ ($1 \leq i \leq n$).

LEMMA 2'. If (0), (I') and (III') hold, there is a $\delta > 0$ such that $s'_{KN} x_K(t) > 0$ on $(b - \delta, b)$.

PROOF. The proof is like that of Lemma 2, using (4). Alternatively, let $A(t) = A(b)$ for $t > b$; let $s = 2b - t$, $B(s) = -A(t)$, and $w(s) = x(t)$; and apply Lemma 2 directly to the system $w'(s) = B(s)w(s)$ on the interval $[b, 2b - a]$.

LEMMA 3'. If (0), (II') and (III') hold, then

$$(6) \quad s'_{iN} x_i(t) > 0 \quad (1 \leq i \leq n)$$

holds on $[a, b)$.

THEOREM 1'. If (II') holds for products excluding the $a_{ii}(t)$'s, the problem

$$(3') \quad y' = Ay + f, \quad Qy(a) + Py(b) = \beta$$

has a unique solution.

Theorem 1 implies a stronger version of Theorem A, since the hypotheses of Theorem A imply (II) for each N . Further, coefficient matrices with vanishing entries can be treated; in particular, known theorems (e.g., see [2] and [3]) for the n th order scalar case are implied.

3. The three-point problem. Let $a < c < b$.

THEOREM 2. Let M and N be fixed ($1 \leq M \leq n$, $1 \leq N \leq n$, $M \neq N$).

Let (II) hold on $[c, b]$ and (II') hold on $[a, c]$ for products excluding the $a_{ii}(t)$'s, for N and also for M , with $s_{MN}s'_{NM} = -1$. Let $R = (\delta_{iM}\delta_{jM})$, $Q = (\delta_{iN}\delta_{jN})$, and $S = E - R - Q$. Then the problem

$$(7) \quad y' = Ay + f, \quad Qy(a) + Sy(c) + Ry(b) = \beta$$

has a unique solution.

PROOF. Let $X' = AX$, $X(c) = E$; it suffices to show that $QX(a) + SX(c) + RX(b)$ is nonsingular, the determinant in question being $x_{NN}(a)x_{MM}(b) - x_{NM}(a)x_{MN}(b)$. By Lemma 3, $x_{MM}(b)$ and $s_{MN}x_{MN}(b)$ are positive; by Lemma 3', $x_{NN}(a)$ and $s'_{NM}x_{NM}(a)$ are positive; hence the determinant is positive.

It is a matter of detail to verify that the hypotheses of Theorem B imply those of Theorem 2. As in the two-point case, coefficient matrices with vanishing entries, and in particular the scalar case (e.g., Theorem 2 in [2]), are allowed.

REFERENCES

1. J. B. Garner and L. P. Burton, *Solutions of linear differential systems satisfying boundary conditions in the large*, Proc. Amer. Math. Soc. **12** (1961), 100-106.
2. J. B. Garner, *Boundary conditions for the linear differential equation*, Amer. Math. Monthly **69** (1962), 47-50.
3. W. Leighton and Z. Nehari, *On the oscillation of solutions of self-adjoint linear differential equations of the fourth order*, Trans. Amer. Math. Soc. **89** (1958), 325-377.

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