# TORSION FREE COVERING MODULES

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Let $A$ be an integral domain and $K$ its field of fractions. An $A$-module $E$ is said to be torsion free if $\alpha x=0$ for $\alpha\in A$, $x\in E$ implies $\alpha=0$ or $x=0$. We will say that a submodule $E_1$ of an $A$-module $E$ is pure in $E$ if $\alpha E_1=\alpha E\cap E_1$ for all $\alpha\in A$. Then if $E$ is torsion free, a submodule $E_1$ of $E$ is pure in $E$ if and only if $E/E_1$ is torsion free. Clearly the union of a chain of pure submodules of a module is still a pure submodule and if $E_2\subset E_1$, are submodules of $E$ such that $E_2$ is pure in $E_1$ and $E_1/E_2$ pure in $E/E_2$ then $E_1$ is pure in $E$.

It is well known that for any $A$-module $E$ there exists a torsion free $A$-module $E_1$ and an epimorphism $p\colon E\to E_1$ such that if $\phi$ is any linear mapping from $E$ into a torsion free module $F$ then there is a unique linear mapping $f\colon E_1\to F$ such that $f\circ p=\phi$, i.e., the diagram

$$\begin{tikzcd} E \arrow[r, "P"] \arrow[dr, "\phi"'] & E_1 \arrow[d, "f"] \\ & F \end{tikzcd}$$

is commutative. It suffices to let $E_1$ be $E/E'$ where $E'$ is the torsion submodule of $E$, i.e., the set of elements of $E$ which are not free and $p$ the canonical mapping $E\to E/E'$.

The object of this paper is to show that for any module $E$ there exists a torsion free $A$-module $T(E)$ and a linear mapping $\psi\colon T(E)\to E$ which is unique "up to isomorphism" subject to the two conditions

(1) the kernel of $\psi$ contains no nontrivial pure submodules of $E$,

(2) if $\phi\colon F\to E$ is a linear mapping where $F$ is torsion free then there is a linear mapping $f\colon F\to T(E)$ such that $\psi\circ f=\phi$.

Such a mapping $\psi$ will be called a torsion free covering of $E$ and $T(E)$ will be called a torsion free covering module of $E$. A linear mapping $\psi\colon E'\to E$ will be said to have the torsion free factor property if for any linear mapping $\phi\colon F\to E$, where $F$ is torsion free there exists a linear mapping $f\colon F\to E'$ such that $\psi\circ f=\phi$.

We first prove four lemmas.

LEMMA 1. *If $\psi\colon E'\to E$ has the torsion free factor property and $E_1$ is a submodule of $E$ then the linear mapping $\psi^{-1}(E_1)\to E_1$ which agrees with $\psi$ on $\psi^{-1}(E_1)$ has the torsion free factor property.*

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PROOF. Trivial.

LEMMA 2. *If $E$ is injective then $\psi: E' \rightarrow E$ has the torsion free factor property if and only if for every linear map $\phi: F \rightarrow E$, where $F$ is torsion free and injective there is a linear mapping $f: F \rightarrow E'$ such that $\psi \circ f = \phi$.*

PROOF. The condition is clearly necessary. If $\phi_1: F_1 \rightarrow E$ is any linear mapping where $F_1$ is torsion free, then since $F_1$ is a submodule of a torsion free injective (hence divisible) module $F$ and since $E$ is injective there exists a linear mapping $\phi: F \rightarrow E$ such that $\phi | F_1 = \phi_1$. Then if $f: F \rightarrow E'$ is such that $\psi \circ f = \phi$ then $\psi \circ (f | E_1) = \phi_1$.

LEMMA 3. *For every module $E$ there exists a torsion free module $E'$ and a linear mapping $\psi: E' \rightarrow E$ having the torsion free factor property.*

PROOF. Using Lemma 1 and the fact that every module is a submodule of an injective module we see that it suffices to assume that $E$ is injective. Then using Lemma 2, we see that in order to prove that a linear mapping $\psi: E' \rightarrow E$ has the torsion free factor property it suffices to show that if $\phi: F \rightarrow E$, where $F$ is torsion free and injective then there is a linear mapping $f: F \rightarrow E'$ such that $\psi \circ f = \phi$.

If we let $E'$ be the direct sum of sufficiently many copies of $K$ then clearly there exists a linear mapping $\psi: E' \rightarrow E$ such that for any linear mapping $\phi': K \rightarrow E$ there is a linear mapping $f': K \rightarrow E'$ such that $\psi \circ f' = \phi'$. Then since any torsion free injective module $F$ is the direct sum of a family of submodules isomorphic to $K$, clearly for any linear mapping $\phi: F \rightarrow E$ there is a linear mapping $f: F \rightarrow E'$ such that $\psi \circ f = \phi$.

LEMMA 4. *If $\psi: E' \rightarrow E$ has the torsion free factor property and $N$ is a submodule of $E'$ contained in the kernel of $\psi$ then the induced mapping $E'/N \rightarrow E$ has the torsion free factor property.*

PROOF. Trivial.

In particular we see that if $\psi: E' \rightarrow E$ has the torsion free factor property where $E'$ is torsion free and $N$ is a maximal element among the pure submodules of $E'$ contained in the kernel of $\psi$ then the induced mapping $E'/N \rightarrow E$ is a torsion free covering of $E$.

This remark coupled with Lemma 3 gives us:

THEOREM 1. *Every module $E$ has a torsion free covering.*

Now we need to show the uniqueness of the torsion free covering.

THEOREM 2. *If $\psi': E' \rightarrow E$ and $\psi'': E'' \rightarrow E$ are two torsion free coverings of $E$ and $f: E' \rightarrow E''$ is a linear mapping such that $\psi'' \circ f = \psi'$ then $f$ is an isomorphism.*

PROOF. Since $\psi''$ is a torsion free covering of $E$ there exists a linear mapping $f: E' \to E''$ such that $\psi'' \circ f = \psi'$. But then the kernel of $f$ is a pure submodule of $E'$ (since $E''$ is torsion free) which is contained in the kernel of $\psi'$. But then since $\psi'$ is a torsion free covering, the kernel of $f$ is 0. Thus $f$ is a monomorphism and so Card $(E') \leqq$ Card $(E'')$. Similarly Card $(E'') \leqq$ Card $(E')$ so Card $(E') =$ Card $(E'')$, i.e., all torsion free coverings of $E$ have the same cardinality. Thus let $X$ be a set containing the elements of $E'$ and $E''$ and such that Card $(X) >$ Card $(E')$. Let $\mathfrak{F}$ be the set of pairs $(E_0, \psi_0)$, where $E_0$ is an $A$-module whose elements are elements of $X$ and where $\psi_0$ is a linear mapping $E_0 \to E$ which is a torsion free covering of $E$. Then $(E', \psi')$ and $(E'', \psi'')$ belong to $\mathfrak{F}$.

Partially order $\mathfrak{F}$ by setting $(E_0, \psi_0) \leqq (E_1, \psi_1)$ if $E_0$ is a submodule of $E_1$ and $\psi_1 | E_0 = \psi_0$. Then $\mathfrak{F}$ has maximal elements for if $\mathfrak{C}$ is a chain of $\mathfrak{F}$ let $E^*$ be the union of the first coordinates of the pairs in $\mathfrak{C}$ with the unique structure of an $A$-module such that $E_0$ is a submodule of $E^*$ for each $(E_0, \psi_0)$ in $\mathfrak{C}$ and let $\psi^*: E^* \to E$ be the unique linear mapping such that $\psi^* | E_0 = \psi_0$ for each pair $(E_0, \psi_0)$ in $\mathfrak{C}$.

Then $\psi^*$ clearly has the torsion free factor property. If $N$ is a pure submodule $E^*$ contained in the kernel of $\psi^*$ then $N \cap E_0$ is a pure submodule of $E_0$ contained in the kernel of $\psi_0$ for each $(E_0, \psi_0)$ in $\mathfrak{C}$. Thus $N \cap E_0 = 0$ for each $(E_0, \psi_0)$ in $\mathfrak{C}$ so $N = 0$. Thus $(E^*, \psi^*)$ belongs to $\mathfrak{F}$. Clearly $(E^*, \psi^*)$ is an upper bound of $\mathfrak{C}$.

Thus assume $(E^*, \psi^*)$ is a maximal element of $\mathfrak{F}$.

Now let $f_1: E^* \to E'$ be any linear mapping such that $\psi' \circ f_1 = \psi^*$. By our previous remarks we know $f_1$ is a monomorphism. We would like to show that it is also an epimorphism. Let $Y \subset X$ be such that Card $(Y) =$ Card $(E' - f_1(E^*))$ and such that $E^* \cap Y = \varnothing$. Such a $Y$ is available since Card $(X) >$ Card $(E') =$ Card $(E^*)$. Let $E_0 = E^* \cup Y$ and let $g$ be a bijection $E_0 \to E'$ such that $g | E^* = f_1$ and $g(Y) = E' - f_1(E^*)$. Then $E_0$ can be made uniquely into an $A$-module so that $g$ becomes an isomorphism. Letting $E_0$ denote this module we see that $E^*$ is a submodule of $E_0$, that $(E_0, \psi' \circ g)$ is an element of $\mathfrak{F}$ and $\psi' \circ g | E^* = \psi' \circ f_1 = \psi^*$ so that $(E^*, \psi^*) \leqq (E_0, \psi' \circ g)$. But $(E^*, \psi^*)$ is a maximal element of $\mathfrak{F}$, hence $Y = \varnothing$ so $E' - f_1(E^*) = \varnothing$ or $f_1$ is an epimorphism. Similarly any linear mapping $f_2: E^* \to E''$ such that $\psi'' \circ f_2 = \psi^*$ is an epimorphism. But $f \circ f_1$ is such a mapping since $\psi'' \circ f \circ f_1 = \psi' \circ f_1 = \psi^*$ hence $f \circ f_1$ is an epimorphism but then $f$ must be an epimorphism. But $f$ is a monomorphism hence an isomorphism. This completes the proof.

Using the fact that the Pontrjagin dual of a compact Abelian group is torsion free if and only if the group is connected [3] we get

COROLLARY. *Every compact Abelian group $G$ can be embedded uniquely up to isomorphism in a connected compact Abelian group $G'$ in such a manner that every continuous homomorphism of $G$ into a connected compact Abelian group can be extended to $G'$ and so that $G'$ has no closed connected proper subgroups containing $G$.*

THEOREM 3. *If $\psi: T(E) \to E$ is a torsion free covering of $E$ with kernel $G$ then the sequence*

$$0 \to \mathrm{Ext}_A^n(F, G) \to \mathrm{Ext}_A^n(F, T(E)) \to \mathrm{Ext}_A^n(F, E) \to 0$$

*is exact if $F$ is torsion free and if $n \geqq 1$.*

PROOF. By definition of $T(E)$, $\mathrm{Hom}(F, T(E)) \to \mathrm{Hom}(F, E) \to 0$ is exact whenever $F$ is torsion free. Choose

$$0 \to K \to L \to F \to 0$$

exact with $L$ a free module. Then $\mathrm{Ext}_A^i(K, \ )$ is naturally isomorphic to $\mathrm{Ext}_A^{i+1}(F, \ )$ for all $i \geqq 1$. Applying the above remarks to $K$, which is torsion free and using induction we get that

$$0 \to \mathrm{Ext}_A^i(F, G) \to \mathrm{Ext}_A^i(F, T(E))$$

is exact for all $i \geqq 1$. Hence

$$0 \to \mathrm{Ext}_A^i(F, G) \to \mathrm{Ext}_A^i(F, T(E)) \to \mathrm{Ext}_A^i(F, E) \to 0$$

is exact for all $i \geqq 1$.

For an example we show:

LEMMA 5. *If $A$ is a principal ring and $\pi$ a prime then if $E$ is a torsion free covering module of the $A$-module $A/(\pi)$ then $E$ is isomorphic to the $\pi$-adic numbers.*

PROOF. It is known that the $\pi$-adic numbers are isomorphic to the inverse limit of the inverse system of $A$-modules defined by the canonical mappings $A/(\pi^{n+1}) \to A/(\pi^n)$, $n = 1, 2, \cdots$. Let $E$ denote this limit and let $\psi: E \to A/(\pi)$ be the projection mapping. It is easy to see that no nontrivial pure submodules of $E$ are contained in the kernel of $\psi$. Let $\phi: F \to A/(\pi)$ be any linear mapping where $F$ is torsion free. If $\phi = 0$ then let $f: F \to E$ be the null mapping. Then $\psi \circ f = \phi$. If $\phi \neq 0$ we choose a base $(\chi_\iota + \pi F)_{\iota \in I}$ of the vector space $F/\pi F$ over $A/(\pi)$ such that for one $\iota_0 \in I$, $\phi(\chi_{\iota_0}) = 1$ and such that $\phi(\chi_\iota) = 0$ for $\iota \neq \iota_0$. Then the family $(\chi_\iota + \pi^n F)_{\iota \in I}$ forms a base of the $A/(\pi^n)$-module $F/\pi^n F$ for first suppose $\sum_{\iota \in I} \alpha_\iota \chi_\iota \in \pi^n F$. Then since

$(\chi_\iota+\pi F)_{\iota\in I}$ is a base of the $A/(\pi)$-module $F/\pi F$ we see that $\pi$ divides each $\alpha_\iota$ and so $\pi$ divides $\sum_{\iota\in I}\alpha_\iota\chi_\iota$. Since $F$ is torsion free the symbol $\sum_{\iota\in I}\alpha_\iota\chi_\iota/\pi$ is well defined and we have $\sum_{\iota\in I}\alpha_\iota\chi_\iota/\pi=\sum_{\iota\in I}\alpha_\iota/\pi\chi_\iota \in\pi^{n-1}F$. Repeating the argument above we see that $\pi^n$ divides $\alpha_\iota$ which says the family $(\chi_\iota+\pi^nF)_{\iota\in I}$ is free over $A/(\pi^n)$. To prove $(\chi_\iota+\pi^nF)_{\iota\in I}$ generates $F/\pi^nF$ remark that since $F$ is torsion free the map $\chi\to\pi^i\chi$ of $F$ onto $\pi^iF$ is an isomorphism which maps $\pi F$ onto $\pi^{i+1}F$. Thus $F/\pi F$ and $\pi^iF/\pi^{i+1}F$ are isomorphic $A$-modules. Thus $(\pi^i\chi_\iota+\pi^{i+1}F)_{\iota\in I}$ is a set of generators of $\pi^iF/\pi^{i+1}F$ for each $i\geqq 1$. But this clearly implies that for any $\chi\in F$ and $i\geqq 1$ we have $\chi-\sum_{\iota\in I}\alpha_\iota\chi_\iota \in\pi^iF$ for some linear combination $\sum_{\iota\in I}\alpha_\iota\chi_\iota$ of the $\chi_\iota$. It is easy to see that no nontrivial pure submodules of $E$ are contained in the kernel of $\psi$. Let $\phi$: $F\to A/(\pi)$ be any linear mapping where $F$ is torsion free. If $\phi=0$ then let $f$: $F\to E$ be the null mapping. Then $\psi\circ f=\phi$. If $\phi\neq 0$ we choose a base $(\chi_\iota+\pi F)_{\iota\in I}$ of the vector space $F/\pi F$ over $A/(\pi)$ such that for one $\iota_0\in I$, $\phi(\chi_{\iota_0})=1$ and such that $\phi(\chi_\iota)=0$ for $\iota\neq\iota_0$. [Clearly the family $(\chi_\iota+\pi^nF)_{\iota\in I}$ forms a base of the $A/(\pi^n)$-module $F/\pi^nF$]. Hence there exists a linear mapping $f_n$: $F/\pi^nF \to A/(\pi^n)$ such that $f_n(\chi_{\iota_0}+\pi^nF)=1+(\pi^n)$ and $f_n(\chi_\iota+\pi^nF)=0$ if $\iota\neq\iota_0$. Passing to the limit we get a mapping $f$: $F\to E$ such that $\psi\circ f=\phi$.

In more generality the torsion free covering modules of simple $A$-modules have the following interesting property.

THEOREM 4. *If $S$ is a simple $A$-module, $\mathfrak{a}\subset A$ is the annihilator of $S$ and $\psi$: $T(S)\to S$ is a torsion free covering of $S$ then $T(S)$ is a direct summand of any module $F$ containing $T(S)$ such that $\mathfrak{a}T(S) =\mathfrak{a}F\cap T(S)$.*

PROOF. Let $j$ be the mapping $T(S)/\mathfrak{a}T(S)\to F/\mathfrak{a}F$ induced by the canonical injection $T(S)\to F$. By hypothesis, $j$ will be an injection. Then since $F/\mathfrak{a}F$ is semi-simple there will be a mapping $\phi_1$: $F/\mathfrak{a}F\to S$ such that $\phi_1\circ j=\psi$. Thus letting $p$ and $p'$ denote the canonical mappings from $T(S)$ into $T(S)/\mathfrak{a}T(S)$ and from $F$ into $F/\mathfrak{a}F$ we get that $\psi=\psi_1\circ p=\phi_1\circ p'\circ i$ where $i$ is the canonical injection $T(S)\to F$. But there exists a linear mapping $f$: $F\to T(S)$ such that $\psi\circ f=\phi_1\circ p'$. Thus $\psi\circ f\circ i=\phi_1\circ p'\circ i=\psi$ hence $f\circ i$ is an automorphism of $T(S)$ by Theorem 2 so that $i(T(S))=T(S)$ is a direct summand of $F$.

We remark that in case $A$ is a principal ideal domain and $S=A/(\pi)$ then the hypothesis $\mathfrak{a}T(S)=T(S)\cap\mathfrak{a}F$ is equivalent to $T(S)$ being a pure submodule of $F$. Then the result of this lemma is a special case of Proposition 2.1, p. 371 of [2].

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## References

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