TORSION FREE COVERING MODULES

EDGAR ENOCHS

Let A be an integral domain and K its field of fractions. An A-module E is said to be torsion free if $\alpha x=0$ for $\alpha\in A$, $x\in E$ implies $\alpha=0$ or $x=0$. We will say that a submodule E_1 of an A-module E is pure in E if $\alpha E_1=\alpha E\cap E_1$ for all $\alpha\in A$. Then if E is torsion free, a submodule E_1 of E is pure in E if and only if E/E_1 is torsion free. Clearly the union of a chain of pure submodules of a module is still a pure submodule and if $E_2\subset E_1$, are submodules of E such that E_2 is pure in E_1 and E_1/E_2 pure in E/E_2 then E_1 is pure in E.

It is well known that for any A-module E there exists a torsion free A-module E_1 and an epimorphism $p: E\to E_1$ such that if ϕ is any linear mapping from E into a torsion free module F then there is a unique linear mapping $f: E_1\to F$ such that $f\circ p=\phi$, i.e., the diagram

$$\begin{tikzcd} E \arrow[r, "P"] \arrow[dr, "\phi"'] & E_1 \arrow[d, "f"] \\ & F \end{tikzcd}$$

is commutative. It suffices to let E_1 be E/E' where E' is the torsion submodule of E, i.e., the set of elements of E which are not free and p the canonical mapping $E\to E/E'$.

The object of this paper is to show that for any module E there exists a torsion free A-module $T(E)$ and a linear mapping ψ: $T(E)\to E$ which is unique "up to isomorphism" subject to the two conditions

(1) the kernel of ψ contains no nontrivial pure submodules of E,

(2) if ϕ: $F\to E$ is a linear mapping where F is torsion free then there is a linear mapping f: $F\to T(E)$ such that $\psi\circ f=\phi$.

Such a mapping ψ will be called a torsion free covering of E and $T(E)$ will be called a torsion free covering module of E. A linear mapping ψ: $E'\to E$ will be said to have the torsion free factor property if for any linear mapping ϕ: $F\to E$, where F is torsion free there exists a linear mapping f: $F\to E'$ such that $\psi\circ f=\phi$.

We first prove four lemmas.

LEMMA 1. *If ψ: $E'\to E$ has the torsion free factor property and E_1 is a submodule of E then the linear mapping $\psi^{-1}(E_1)\to E_1$ which agrees with ψ on $\psi^{-1}(E_1)$ has the torsion free factor property.*

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PROOF. Trivial.

LEMMA 2. *If E is injective then $\psi: E' \to E$ has the torsion free factor property if and only if for every linear map $\phi: F \to E$, where F is torsion free and injective there is a linear mapping $f: F \to E'$ such that $\psi \circ f = \phi$.*

PROOF. The condition is clearly necessary. If $\phi_1: F_1 \to E$ is any linear mapping where F_1 is torsion free, then since F_1 is a submodule of a torsion free injective (hence divisible) module F and since E is injective there exists a linear mapping $\phi: F \to E$ such that $\phi | F_1 = \phi_1$. Then if $f: F \to E'$ is such that $\psi \circ f = \phi$ then $\psi \circ (f | E_1) = \phi_1$.

LEMMA 3. *For every module E there exists a torsion free module E' and a linear mapping $\psi: E' \to E$ having the torsion free factor property.*

PROOF. Using Lemma 1 and the fact that every module is a submodule of an injective module we see that it suffices to assume that E is injective. Then using Lemma 2, we see that in order to prove that a linear mapping $\psi: E' \to E$ has the torsion free factor property it suffices to show that if $\phi: F \to E$, where F is torsion free and injective then there is a linear mapping $f: F \to E'$ such that $\psi \circ f = \phi$.

If we let E' be the direct sum of sufficiently many copies of K then clearly there exists a linear mapping $\psi: E' \to E$ such that for any linear mapping $\phi': K \to E$ there is a linear mapping $f': K \to E'$ such that $\psi \circ f' = \phi'$. Then since any torsion free injective module F is the direct sum of a family of submodules isomorphic to K, clearly for any linear mapping $\phi: F \to E$ there is a linear mapping $f: F \to E'$ such that $\psi \circ f = \phi$.

LEMMA 4. *If $\psi: E' \to E$ has the torsion free factor property and N is a submodule of E' contained in the kernel of ψ then the induced mapping $E'/N \to E$ has the torsion free factor property.*

PROOF. Trivial.

In particular we see that if $\psi: E' \to E$ has the torsion free factor property where E' is torsion free and N is a maximal element among the pure submodules of E' contained in the kernel of ψ then the induced mapping $E'/N \to E$ is a torsion free covering of E.

This remark coupled with Lemma 3 gives us:

THEOREM 1. *Every module E has a torsion free covering.*

Now we need to show the uniqueness of the torsion free covering.

THEOREM 2. *If $\psi': E' \to E$ and $\psi'': E'' \to E$ are two torsion free coverings of E and $f: E' \to E''$ is a linear mapping such that $\psi'' \circ f = \psi'$ then f is an isomorphism.*

PROOF. Since ψ'' is a torsion free covering of E there exists a linear mapping $f: E' \to E''$ such that $\psi'' \circ f = \psi'$. But then the kernel of f is a pure submodule of E' (since E'' is torsion free) which is contained in the kernel of ψ'. But then since ψ' is a torsion free covering, the kernel of f is 0. Thus f is a monomorphism and so Card $(E') \leqq$ Card (E''). Similarly Card $(E'') \leqq$ Card (E') so Card $(E') =$ Card (E''), i.e., all torsion free coverings of E have the same cardinality. Thus let X be a set containing the elements of E' and E'' and such that Card (X) $>$ Card (E'). Let $\mathfrak{F}$ be the set of pairs (E_0, ψ_0), where E_0 is an A-module whose elements are elements of X and where ψ_0 is a linear mapping $E_0 \to E$ which is a torsion free covering of E. Then (E', ψ') and (E'', ψ'') belong to $\mathfrak{F}$.

Partially order $\mathfrak{F}$ by setting $(E_0, \psi_0) \leqq (E_1, \psi_1)$ if E_0 is a submodule of E_1 and $\psi_1 | E_0 = \psi_0$. Then $\mathfrak{F}$ has maximal elements for if $\mathfrak{C}$ is a chain of $\mathfrak{F}$ let E^* be the union of the first coordinates of the pairs in $\mathfrak{C}$ with the unique structure of an A-module such that E_0 is a submodule of E^* for each (E_0, ψ_0) in $\mathfrak{C}$ and let $\psi^*: E^* \to E$ be the unique linear mapping such that $\psi^* | E_0 = \psi_0$ for each pair (E_0, ψ_0) in $\mathfrak{C}$.

Then ψ^* clearly has the torsion free factor property. If N is a pure submodule E^* contained in the kernel of ψ^* then $N \cap E_0$ is a pure submodule of E_0 contained in the kernel of ψ_0 for each (E_0, ψ_0) in $\mathfrak{C}$. Thus $N \cap E_0 = 0$ for each (E_0, ψ_0) in $\mathfrak{C}$ so $N = 0$. Thus (E^*, ψ^*) belongs to $\mathfrak{F}$. Clearly (E^*, ψ^*) is an upper bound of $\mathfrak{C}$.

Thus assume (E^*, ψ^*) is a maximal element of $\mathfrak{F}$.

Now let $f_1: E^* \to E'$ be any linear mapping such that $\psi' \circ f_1 = \psi^*$. By our previous remarks we know f_1 is a monomorphism. We would like to show that it is also an epimorphism. Let $Y \subset X$ be such that Card $(Y) =$ Card $(E' - f_1(E^*))$ and such that $E^* \cap Y = \varnothing$. Such a Y is available since Card $(X) >$ Card $(E') =$ Card (E^*). Let $E_0 = E^* \cup Y$ and let g be a bijection $E_0 \to E'$ such that $g | E^* = f_1$ and $g(Y) = E' - f_1(E^*)$. Then E_0 can be made uniquely into an A-module so that g becomes an isomorphism. Letting E_0 denote this module we see that E^* is a submodule of E_0, that $(E_0, \psi' \circ g)$ is an element of $\mathfrak{F}$ and $\psi' \circ g | E^* = \psi' \circ f_1 = \psi^*$ so that $(E^*, \psi^*) \leqq (E_0, \psi' \circ g)$. But (E^*, ψ^*) is a maximal element of $\mathfrak{F}$, hence $Y = \varnothing$ so $E' - f_1(E^*) = \varnothing$ or f_1 is an epimorphism. Similarly any linear mapping $f_2: E^* \to E''$ such that $\psi'' \circ f_2 = \psi^*$ is an epimorphism. But $f \circ f_1$ is such a mapping since $\psi'' \circ f \circ f_1 = \psi' \circ f_1 = \psi^*$ hence $f \circ f_1$ is an epimorphism but then f must be an epimorphism. But f is a monomorphism hence an isomorphism. This completes the proof.

Using the fact that the Pontrjagin dual of a compact Abelian group is torsion free if and only if the group is connected [3] we get

COROLLARY. *Every compact Abelian group G can be embedded uniquely up to isomorphism in a connected compact Abelian group G' in such a manner that every continuous homomorphism of G into a connected compact Abelian group can be extended to G' and so that G' has no closed connected proper subgroups containing G.*

THEOREM 3. *If ψ: $T(E) \to E$ is a torsion free covering of E with kernel G then the sequence*

$$0 \to \operatorname{Ext}_A^n (F, G) \to \operatorname{Ext}_A^n (F, T(E)) \to \operatorname{Ext}_A^n (F, E) \to 0$$

is exact if F is torsion free and if $n \geqq 1$.

PROOF. By definition of $T(E)$, $\operatorname{Hom}(F, T(E)) \to \operatorname{Hom}(F, E) \to 0$ is exact whenever F is torsion free. Choose

$$0 \to K \to L \to F \to 0$$

exact with L a free module. Then $\operatorname{Ext}_A^i(K, \ )$ is naturally isomorphic to $\operatorname{Ext}_A^{i+1}(F, \ )$ for all $i \geqq 1$. Applying the above remarks to K, which is torsion free and using induction we get that

$$0 \to \operatorname{Ext}_A^i (F, G) \to \operatorname{Ext}_A^i (F, T(E))$$

is exact for all $i \geqq 1$. Hence

$$0 \to \operatorname{Ext}_A^i (F, G) \to \operatorname{Ext}_A^i (F, T(E)) \to \operatorname{Ext}_A^i (F, E) \to 0$$

is exact for all $i \geqq 1$.

For an example we show:

LEMMA 5. *If A is a principal ring and π a prime then if E is a torsion free covering module of the A-module $A/(\pi)$ then E is isomorphic to the π-adic numbers.*

PROOF. It is known that the π-adic numbers are isomorphic to the inverse limit of the inverse system of A-modules defined by the canonical mappings $A/(\pi^{n+1}) \to A/(\pi^n)$, $n = 1, 2, \cdots$. Let E denote this limit and let ψ: $E \to A/(\pi)$ be the projection mapping. It is easy to see that no nontrivial pure submodules of E are contained in the kernel of ψ. Let ϕ: $F \to A/(\pi)$ be any linear mapping where F is torsion free. If $\phi = 0$ then let f: $F \to E$ be the null mapping. Then $\psi \circ f = \phi$. If $\phi \neq 0$ we choose a base $(\chi_\iota + \pi F)_{\iota \in I}$ of the vector space $F/\pi F$ over $A/(\pi)$ such that for one $\iota_0 \in I$, $\phi(\chi_{\iota_0}) = 1$ and such that $\phi(\chi_\iota) = 0$ for $\iota \neq \iota_0$. Then the family $(\chi_\iota + \pi^n F)_{\iota \in I}$ forms a base of the $A/(\pi^n)$-module $F/\pi^n F$ for first suppose $\sum_{\iota \in I} \alpha_\iota \chi_\iota \in \pi^n F$. Then since

$(\chi_\iota+\pi F)_{\iota\in I}$ is a base of the $A/(\pi)$-module $F/\pi F$ we see that π divides each α_ι and so π divides $\sum_{\iota\in I}\alpha_\iota\chi_\iota$. Since F is torsion free the symbol $\sum_{\iota\in I}\alpha_\iota\chi_\iota/\pi$ is well defined and we have $\sum_{\iota\in I}\alpha_\iota\chi_\iota/\pi=\sum_{\iota\in I}\alpha_\iota/\pi\chi_\iota \in\pi^{n-1}F$. Repeating the argument above we see that π^n divides α_ι which says the family $(\chi_\iota+\pi^nF)_{\iota\in I}$ is free over $A/(\pi^n)$. To prove $(\chi_\iota+\pi^nF)_{\iota\in I}$ generates F/π^nF remark that since F is torsion free the map $\chi\rightarrow\pi^i\chi$ of F onto π^iF is an isomorphism which maps πF onto $\pi^{i+1}F$. Thus $F/\pi F$ and $\pi^iF/\pi^{i+1}F$ are isomorphic A-modules. Thus $(\pi^i\chi_\iota+\pi^{i+1}F)_{\iota\in I}$ is a set of generators of $\pi^iF/\pi^{i+1}F$ for each $i\geqq 1$. But this clearly implies that for any $\chi\in F$ and $i\geqq 1$ we have $\chi-\sum_{\iota\in I}\alpha_\iota\chi_\iota \in\pi^iF$ for some linear combination $\sum_{\iota\in I}\alpha_\iota\chi_\iota$ of the χ_ι. It is easy to see that no nontrivial pure submodules of E are contained in the kernel of ψ. Let ϕ: $F\rightarrow A/(\pi)$ be any linear mapping where F is torsion free. If $\phi=0$ then let f: $F\rightarrow E$ be the null mapping. Then $\psi\circ f=\phi$. If $\phi\neq 0$ we choose a base $(\chi_\iota+\pi F)_{\iota\in I}$ of the vector space $F/\pi F$ over $A/(\pi)$ such that for one $\iota_0\in I$, $\phi(\chi_{\iota_0})=1$ and such that $\phi(\chi_\iota)=0$ for $\iota\neq\iota_0$. [Clearly the family $(\chi_\iota+\pi^nF)_{\iota\in I}$ forms a base of the $A/(\pi^n)$-module F/π^nF]. Hence there exists a linear mapping f_n: $F/\pi^nF \rightarrow A/(\pi^n)$ such that $f_n(\chi_{\iota_0}+\pi^nF)=1+(\pi^n)$ and $f_n(\chi_\iota+\pi^nF)=0$ if $\iota\neq\iota_0$. Passing to the limit we get a mapping f: $F\rightarrow E$ such that $\psi\circ f=\phi$.

In more generality the torsion free covering modules of simple A-modules have the following interesting property.

THEOREM 4. *If S is a simple A-module, $\mathfrak{a}\subset A$ is the annihilator of S and ψ: $T(S)\rightarrow S$ is a torsion free covering of S then $T(S)$ is a direct summand of any module F containing $T(S)$ such that* $\mathfrak{a}T(S)=\mathfrak{a}F\cap T(S)$.

PROOF. Let j be the mapping $T(S)/\mathfrak{a}T(S)\rightarrow F/\mathfrak{a}F$ induced by the canonical injection $T(S)\rightarrow F$. By hypothesis, j will be an injection. Then since $F/\mathfrak{a}F$ is semi-simple there will be a mapping ϕ_1: $F/\mathfrak{a}F\rightarrow S$ such that $\phi_1\circ j=\psi$. Thus letting p and p' denote the canonical mappings from $T(S)$ into $T(S)/\mathfrak{a}T(S)$ and from F into $F/\mathfrak{a}F$ we get that $\psi=\psi_1\circ p=\phi_1\circ p'\circ i$ where i is the canonical injection $T(S)\rightarrow F$. But there exists a linear mapping f: $F\rightarrow T(S)$ such that $\psi\circ f=\phi_1\circ p'$. Thus $\psi\circ f\circ i=\phi_1\circ p'\circ i=\psi$ hence $f\circ i$ is an automorphism of $T(S)$ by Theorem 2 so that $i(T(S))=T(S)$ is a direct summand of F.

We remark that in case A is a principal ideal domain and $S=A/(\pi)$ then the hypothesis $\mathfrak{a}T(S)=T(S)\cap\mathfrak{a}F$ is equivalent to $T(S)$ being a pure submodule of F. Then the result of this lemma is a special case of Proposition 2.1, p. 371 of [2].

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UNIVERSITY OF SOUTH CAROLINA