UNIQUE FACTORIZATION OF IDEALS INTO NONFACTORABLE IDEALS

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The purpose of this note is to prove a theorem which shows a connection between the definition of a prime ideal in classical algebraic number theory and the usual definition of a prime ideal. A proper ideal in an integral domain with unit element is an ideal different from the unit ideal and the zero ideal. An ideal A will be called nonfactorable provided A is a proper ideal and A = BC (where B and C are ideals) implies that either B or C is the unit ideal.

Theorem. If J is an integral domain with unit such that every proper ideal of J is either a nonfactorable ideal or can be factored uniquely into a product of nonfactorable ideals, then J is a Dedekind domain and the nonfactorable ideals are prime in the usual sense.

Proof. Let P be a proper prime ideal of J and p ≠ 0 be an element of P. There exist nonfactorable ideals N₁, · · · , Nₙ in J such that (p) = N₁· · · · · · Nₙ. Let Nᵢ be an arbitrary member of the collection N₁, · · · , Nₙ. Since (p) is an invertible ideal, it follows that Nᵢ is an invertible ideal and consequently Nᵢ is finitely generated (see [1, p. 272]).

Let x be any element of J such that x is not an element of Nᵢ. Since Nᵢ is finitely generated, then Nᵢ+(x) is finitely generated. The cancellation law for ideals is valid in J (an obvious consequence of the unique factorization property) and therefore finitely generated ideals are invertible (see [2, p. 13]). Hence Nᵢ+(x) is invertible and since Nᵢ+(x) ⊇ Nᵢ, there exists an ideal Q in J such that [Nᵢ+(x)]·Q = Nᵢ. It is clear that Nᵢ+(x) = J and Nᵢ is a maximal ideal.

Since P is a prime ideal and P ⊇ N₁· · · · · · Nₙ it follows that P ⊇ Nᵢ for some i and therefore P is invertible. Hence every proper prime ideal of J is invertible and J is a Dedekind domain (see [3, Theorem 7, p. 33]).

References


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