SOME RELATIONSHIPS BETWEEN LOCALLY SUPER-ADDITIVE FUNCTIONS AND CONVEX FUNCTIONS[1]

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1. Introduction. It is well known that a real valued function f defined on an interval I is convex if and only if it is locally convex; i.e., each point in I has a neighborhood on which f is convex. In this paper we examine a condition which we call local superadditivity. This condition is weaker than convexity. However, as we shall see, under suitable regularity hypotheses, a function which is locally superadditive must be convex. We shall also show that slight weakenings in these regularity hypotheses result in spectacular failure of the conclusion.

2. **Preliminaries.** In this section we define the terms we shall be using. We present these in the form most suitable for our applications.

DEFINITION 2.1. The function f defined on an interval I is called *convex* provided that given any two subintervals of equal length, $[a, a+h]$ and $[b, b+h]$ with $b>a$, then $f(b+h)-f(b) \geqq f(a+h)-f(a)$. For a long reference list on convex functions and related topics, see Beckenbach [1].

DEFINITION 2.2. The function f defined on an interval $I=[a, b]$ is called *superadditive on the interval I* provided that for any positive $h<b-a$ and $x \in [a, b-h]$, $f(a+h)-f(a) \leqq f(x+h)-f(x)$. This is a modification of the usual notion of a superadditive function, according to which f is superadditive on S if $f(x+y) \geqq f(x)+f(y)$ whenever x, y and $x+y \in S$. (See [2, p. 1155]; [7].) However, a function f defined on $[0, a]$ and vanishing at the origin is superadditive on the interval $[0, a]$ if and only if it is superadditive. Our definition obviates the special consideration which must be given to the origin in the usual theory of superadditive functions (a translation of a superadditive function need not be superadditive), and also eliminates trivially superadditive functions (any function f is superadditive on $[a, b]$ if $0<a<b<2a$). Functions superadditive on the interval $[0, a]$ have been studied by Laatsch [6] and by the author [2; 3]. In the sequel we shall use the notation SA to mean superadditive on the interval.

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The calculations in connection with the Cantor function and the function of Theorem 3.3 were carried out by John Leonard.

We see from the above definitions that a function defined on $[a, b]$ is convex provided it increases no less on a given interval in its domain than it does on any "earlier" interval of the same length, and it is SA provided it increases no less on a given interval of its domain than it does on the "initial" interval of the same length. Thus one can look upon the theory of SA functions as a generalization of the theory of convex functions.

DEFINITION 2.3. The function f is called *locally superadditive* on an interval I provided for every $x_0 \in I$, there exist arbitrarily small neighborhoods of x_0 on which f is SA.

3. Comparison of local superadditivity and convexity. We now proceed to show the relationships between local superadditivity and convexity.

THEOREM 3.1. *Let f be locally superadditive and differentiable on I, with f' continuous a.e. Then f is convex.*

PROOF. We shall show that f' is increasing. Let x_0 be any point of continuity of f' and let $y > x_0$, $y \in I$. For each $x \in [x_0, y]$ there exists an interval I_x containing x such that f is SA on I_x. It follows from the Heine-Borel theorem that a finite number of these intervals cover $[x_0, y]$. Denote these intervals by $[a_0, b_0]$, $[a_1, b_1], \cdots, [a_N, b_N]$, where $a_i < a_{i+1} < b_i < b_{i+1}$, $i = 0, \cdots, N-1$. Since f is SA on $[a_i, b_i]$, we have min $\{f'(x): a_i \leqq x \leqq b_i\} = f'(a_i)$, for an SA function attains its minimum derivative at the left hand endpoint of the interval of definition [5, p. 251]. Thus

$$f'(y) \geqq f'(a_N) \geqq f'(a_{N-1}) \geqq \cdots \geqq f'(a_0). \tag{1}$$

Since the interval about x_0 on which f is SA can be taken as small as we like, a_0 can be chosen arbitrarily close to x_0. It follows from (1) and the continuity of f' at x_0 that $f'(y) \geqq f'(x_0)$.

We have shown that if f' is continuous at x_0 and $y > x_0$, then $f'(y) \geqq f'(x_0)$. Now suppose there exists $y \in I$, $y < x_0$, but $f'(y) > f'(x_0)$. Let $E = \{x: y < x < x_0,\ f'(y) > f'(x) > f'(x_0)\}$. The set E has positive Lebesgue measure. (See Denjoy [4, p. 320].) Since f' is continuous a.e., there is a point $x \in E$ at which f' is continuous. Since $x \in E$, $f'(x) > f'(x_0)$, but since f' is continuous at x, $f'(x) \leqq f'(x_0)$, a contradiction. The conclusion follows now from the fact that the set of points of continuity of f' is dense in I.

If local superadditivity is replaced by superadditivity, then we cannot conclude convexity, even on any interval, as is shown in Theorem 3.2.

LEMMA. *If F is superadditive on $[0, a]$ and G is an increasing nonnegative function, then FG is superadditive.*

The proof is straightforward and will be omitted.

THEOREM 3.2. *There exists a continuously differentiable superadditive function u which is not convex on any interval.*

PROOF. Let p be a positive, continuous, nowhere differentiable function defined on $[0, 1]$, and let g be defined by $g(x) = \int_0^x p(t)dt$. Then g is increasing and continuously differentiable. Let u be defined by the equation $u(x) = xg(x)$. By the lemma, u is superadditive. We have $u'(x) = xg'(x) + g(x)$ for all $x \in [0, 1]$. The function u' is continuous but nowhere differentiable in $(0, 1]$ and hence is everywhere oscillating. It follows that u is not convex (nor concave) on any interval.

One can easily verify that a function defined on $[0, a]$ cannot be locally superadditive and simultaneously subadditive without being linear. It is interesting to note, however, that the Cantor function is subadditive (see Laatsch [6, p. 42]), but fails to be locally superadditive only because there is no neighborhood of 0 on which it is SA. To see this, let c be the Cantor function and let P be the Cantor discontinuum. Given $\delta > 0$ and $x_0 \neq 0$, we must choose a neighborhood of x_0, having length less than δ, on which c is SA. If x_0 is in or is a right hand endpoint of a complementary interval of P, the constancy of c on the interval makes the choice trivial. If x_0 is a left hand endpoint of a complementary interval, choose n such that $1/3^n < \delta$, and let the neighborhood of x_0 be $(x_0 - 2/3^{n+1}, x_0 + 1/3^{n+3})$. If x_0 is none of the above, it is a limit point of endpoints of complementary intervals of P, and hence has a ternary expansion containing infinitely many twos, infinitely many zeroes, and no ones. Represent this expansion as $x_0 = 0.a_1a_2a_3 \cdots$. Choose m such that $1/3^m < \delta$. Choose $n > m$ such that $a_n = 2$. Then c is easily shown to be SA on the interval $(0.a_1a_2 \cdots a_{n-1}0222 \cdots, 0.a_1a_2 \cdots a_{n-1}222 \cdots)$, which contains x_0.

Now let ϵ be an arbitrary positive number and define a function c^* by the equations

$$c^*(x) = \begin{cases} 0 & 0 \leqq x \leqq \epsilon \\ c(x - \epsilon) & \epsilon < x \leqq 1 + \epsilon. \end{cases}$$

The function c^* is locally superadditive, differentiable a.e., and its derivative is continuous wherever it exists. It is, of course, not convex. This should be compared with the hypothesis of Theorem 3.1.

An example of a superadditive function which is not locally superadditive is furnished by any superadditive function which is strictly concave on some interval. The next theorem shows that a function can be locally superadditive without being SA (or, therefore, convex), even among the class of functions satisfying a Lipschitz condition.

THEOREM 3.3. *There exists a function w which is locally superadditive and which satisfies a (uniform) Lipschitz condition, but which is not SA.*

Let K be the nowhere dense perfect set of measure 4/7 obtained from $[0, 1]$ by removing "middle" open intervals of length $1/3^{2n-1}$, $n=1, 2, \cdots$. Let $h(x)=(2r-1)/2^n$ for x in the rth removed interval of length $1/3^{2n-1}$, $1 \leqq r \leqq 2^{n-1}$; $h(0)=0$; $h(1)=1$. Let $v(x)=\sup \{h(t): t<x, t \in [0, 1]-K\}$. Finally, let ϵ be any number satisfying $0<\epsilon<1/3$. Define a function w by the equations

$$w(x) = \begin{cases} 0 & 0 \leqq x \leqq \epsilon \\ v(x-\epsilon) & \epsilon < x \leqq 1+\epsilon. \end{cases}$$

The function w is continuous and nondecreasing, and can be shown to be locally superadditive by a method similar to that used with the function c above. Utilizing the continuity of w and the denseness of the removed intervals in K, one can show that the Dini derivates of w are bounded by 0 and 7/4, so w satisfies a Lipschitz condition. Yet w is not SA, since $w(2/3+\epsilon)-w(1/3+\epsilon)=1/2-1/2=0$, while $w(1/3+\epsilon)-w(\epsilon)=1/2$.

The differentiability requirement can be dropped from Theorem 3.1 if the notion of local superadditivity is strengthened.

DEFINITION 3.1. The function f is *strongly locally superadditive* on I provided for every x_0 in the interior of I there exist arbitrarily small neighborhoods, centered at x_0, on which f is SA.

THEOREM 3.4. *If f is continuous and strongly locally superadditive on I, then f is convex on I.*

PROOF. Suppose f is not convex on I. Then there exist $x_1, x_2 \in I$ such that $l_{12}(x) \leqq f(x)$, all $x \in [x_1, x_2]$, where l_{12} denotes the linear function satisfying $l(x_i)=f(x_i)$, $i=1, 2$. Let x_0 be a point in (x_1, x_2) such that $f(x_0)/x_0 \geqq f(x)/x$ for all $x \in (x_1, x_2)$ and for which *strict* inequality holds on some interval having x_0 as right hand endpoint, say $[x_3, x_0]$. Let $[a, b]$ be an interval having x_0 as midpoint such that $x_3<a<x_0<b<x_2$. It is clear that $f(x_0)-f(a)>f(b)-f(x_0)$, so that f cannot be SA on $[a, b]$. This contradicts the local superadditivity of f.

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ON ERGODIC MEASURES

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Introduction, notation, definitions and known results. In what follows Ω will be a compact metric space and T a homeomorphism of Ω onto itself. The pair (Ω, T) is called a compact discrete dynamical system. If m is a Borel measure in Ω such that $m(\Omega)=1$, m is called a normalised Borel measure. Throughout this paper all measures will be assumed to be normalised Borel measures. If for any Borel set B, $m(B)=m(TB)$, m is called an invariant measure. Ergodic invariant measures will simply be referred to as ergodic measures. If, for some set A, $T(A)=A$, A will be called an invariant set. A nonempty, closed, invariant set which has no proper subset of the same properties is called a minimal set.

The empty set will be denoted by $\emptyset$. Given two sets A and B, '$A \subseteq B$' will stand for 'A is a subset of B', while '$A \subset B$' for 'A is a proper subset of B'.

The set of points p, $T(p)$, $T^2(p)$, $\cdots$ is called the positive semiorbit of p and will be denoted by O_p^+. The set of limit points of the sequence p, $T(p)$, $T^2(p)$, $\cdots$ in Ω is called the ω-limit set of p and will be denoted by ω_p. The set of points p, $T(p)$, $T^{-1}(p)$, $T^2(p)$, $T^{-2}(p)$, $\cdots$ is called the orbit of p and will be denoted by O_p. The closure of the semiorbit (orbit) in Ω is called the semiorbit closure (orbit closure) and will be denoted by $\overline{O}_p^+$ $(\overline{O}_p)$. $\mathfrak{N}(p, \rho)$ will denote the open sphere about p of radius ρ.

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