# ON COMPACT COMPLEX COSET SPACES OF REDUCTIVE LIE GROUPS

JUN-ICHI HANO[1]

**1. The statement of theorems.** Let $G$ be a connected complex Lie group and let $B$ be a closed complex Lie subgroup in $G$. The left coset space $G/B$ is a complex manifold, which will be called a complex coset space. We denote by $B_0$ the identity connected component of $B$ and $U$ the normalizer of $B_0$. The canonical projection $p$ of $G/B$ onto $G/U$ defines a holomorphic fibre bundle, and the complex Lie group $U/B_0$ acts on $G/B$ as the structure group. We denote by $(G/B, p, G/U)$ this holomorphic fibre bundle.

Suppose that the complex coset space $G/B$ is compact. Then, by a recent result of Borel-Remmert [1, Satz 7′], it turns out that the base space $G/U$ is a Kaehler $C$-space, that is, a simply connected compact complex coset space admitting a Kaehler metric, such that the group of isometries is transitive on it. Since $G/U$ is simply connected, $U$ must be connected. The complex coset space of the connected complex Lie group $U/B_0$ by the discrete subgroup $B/B_0$ can be regarded as the standard fibre of $(G/B, p, G/U)$. Making use of this result of Borel-Remmert we derive the following.

THEOREM 1. *Let $G/B$ be a compact connected complex coset space of a connected complex Lie group $G$ by a closed complex Lie subgroup $B$. Let $U$ be the normalizer of the identity connected component $B_0$ of $B$. If $G$ is a reductive complex Lie group, then the fibre of the holomorphic fibre bundle $(G/B, p, G/U)$ is a compact connected complex coset space of a reductive complex Lie group $U/B_0$ by the discrete subgroup $B/B_0$.*

If a compact coset space of a connected reductive real Lie group $G'$ by a closed Lie subgroup $B'$ admits an invariant complex structure, the complex manifold $G'/B'$ can be written as a complex coset space of a connected complex reductive Lie group. Hence, this is a case where we can apply the above theorem. This gives a generalization of a theorem proved by Matsushima [2, Theorem 2].

Let $M$ be a connected compact complex manifold. By a theorem of Bochner-Montgomery, the group of all holomorphic homeomorphisms of $M$ onto itself is a complex Lie group acting on $M$ as a holomorphic transformation group. We denote by $A_0(M)$ the identity

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connected component of this complex Lie group. If the group $A_0(M)$ acts on $M$ transitively, $M$ can be expressed as a complex coset space of a connected complex Lie group $A_0(M)$.

THEOREM 2. *Let $M$ be a connected compact coset space, and let $A_0(M)$ be the identity connected component of the complex Lie group of holomorphic homeomorphisms of $M$ onto itself. If a connected reductive (real or complex) Lie subgroup in $A_0(M)$ is transitive on $M$, then $A_0(M)$ is a complex reductive Lie group.*

When $M$ is a $C$-space, that is, a simply connected compact complex coset space, the fact that $A_0(M)$ is a reductive complex Lie group, more strongly that, it is locally a direct product of a complex vector group and a connected semi-simple complex Lie group, is proved by Wang [**3**, Theorem III].

2. **The proof of Theorem 1.** We shall prove Theorem 1 in a slightly more general form, which is required in the proof of Theorem 2. Using the same notations as in Theorem 1, let $X$ be a closed complex subgroup in $G$ such that $B \subset X \subset U$. We shall show that if the complex coset space $G/X$ is a Kaehler $C$-space, then the factor group $X/B_0$ is reductive. By the theorem of Borel-Remmert mentioned above, the subgroup $U$ satisfies the hypothesis for the group $X$.

Wang's structure theorem asserts that a complex coset space of a connected complex Lie group $G$ by a closed complex Lie subgroup $X$ is a Kaehler $C$-space if and only if $X$ is connected and contains a maximal connected solvable Lie subgroup in $G$ [**3**]. Therefore, $X$ is connected and contains the identity connected component $Z$ of the center of $G$. We denote by $g$ the Lie algebra of $G$ and by $u$, $x$, $b$, and $z$ the subalgebras in $g$ corresponding to the complex Lie subgroups $U$, $X$, $B_0$, and $Z$, respectively. We always understand that the base field of those Lie algebras is the field of complex numbers. As $G$ is reductive, $g$ is the direct sum of the center $z$ and the maximal semi-simple ideal $g_1$. We review here a proposition by Wang [**3**, Proposition 5.2] about the subalgebra $x$ corresponding to the isotropy subgroup $X$ of the Kaehler $C$-space $G/X$. We can choose a Cartan subalgebra $h$ of $g_1$ which is contained in $x$, an ordering of the set of roots with respect to $h$, and a subset $\Delta'$ consisting of some positive roots, so that a root vector $X_\alpha$ belonging to a root $\alpha$ is contained in $x$ if and only if either $\alpha$ is positive or $-\alpha$ is in $\Delta'$. Let $\Delta''$ be the set of the positive roots not contained in $\Delta'$. Then, we have

$$x = z + h + \sum_{\pm\alpha \in \Delta'} \{X_\alpha\} + \sum_{\beta \in \Delta''} \{X_\beta\}.$$

The subspace $n$ spanned by the $X_\beta$, $\beta \in \Delta''$, is a nilpotent ideal in $x$, and a factor algebra of $x$ by an ideal containing $n$ is always a reductive Lie algebra.

Thus, in order to complete the proof of the statement, it suffices to show that $b \supset n$. Since $b$ is an ideal in $x$, $x$ is stable by ad $H$, $H \in h$. This implies that $b$ is spanned by $b \cap (z+h)$ and some root vectors. First, we shall show that for a root $\alpha$ in $\Delta'$, $X_\alpha$ belongs to $b$ if and only if $X_{-\alpha}$ belongs to $b$. Take a root vector $X_\alpha$ such that $X_\alpha \in b$ and that either $\alpha$ or $-\alpha$ belongs to $\Delta'$. Then, $[X_{-\alpha}, X_\alpha] \in b \cap h$ and $[[X_{-\alpha}, X_\alpha], X_{-\alpha}] = \alpha([X_{-\alpha}, X_\alpha]) \cdot X_{-\alpha} \in b$. As is known in the theory of a semi-simple complex Lie algebra, the complex number $\alpha([X_{-\alpha}, X_\alpha])$ never vanishes, which implies that $X_{-\alpha} \in b$. Let us denote by $\mathrm{ad}_b\, H$ and $\mathrm{ad}_x\, H$ the restrictions of ad $H$ to the subspaces $b$ and $x$, respectively, and by $\bar{\Delta}$ the set of the roots $\alpha$ such that $\alpha \in \Delta''$ and that $X_\alpha \in b$. Then, from the fact shown above it follows that for any $H \in h$,

$$\mathrm{trace}\,(\mathrm{ad}_x\, H) - \mathrm{trace}\,(\mathrm{ad}_b\, H) = \sum_{\alpha \in \bar{\Delta}} \alpha(H).$$

Since the coset space $X/B$ of $X/B_0$ by the discrete subgroup $B/B_0$ is compact, the factor group $X/B_0$ is a unimodular Lie group. Therefore, the trace of the linear transformation ad $H$, $H \in h$, must vanish, and accordingly the sum of all roots in $\bar{\Delta}$ is equal to zero. On the other hand, as every root in $\bar{\Delta}$ is positive, we see that the set $\bar{\Delta}$ is empty. This implies that $b$ contains $n$, completing the proof.

3. **The proof of Theorem 2.** We denote by $\overline{G}$, $\overline{B}$ the group $A_0(M)$, its isotropy subgroup at a point in $M$, respectively. Let $\overline{U}$ be the normalizer of the identity component $\overline{B}_0$ of $\overline{B}$. In virtue of the theorem of Borel-Remmert mentioned above, $\overline{G}/\overline{U}$ is a Kaehler $C$-space and $\overline{U}$ is connected. We denote by $\bar{p}$ the canonical projection of $\overline{G}/\overline{B}$ onto $\overline{G}/\overline{U}$.

Let $G$ be the least connected complex Lie subgroup in $\overline{G}$ containing the given connected reductive Lie subgroup which is transitive on $\overline{G}/\overline{B}$. Obviously, $G$ is also reductive and is transitive on $\overline{G}/\overline{B}$. The isotropy subgroup $B$ in $G$ is $G \cap \overline{B}$. The group $G$ acts on $\overline{G}/\overline{U}$ transitively and its isotropy subgroup $X$ is equal to $G \cap \overline{U}$. Since $\overline{U}$ is the normalizer of $\overline{B}_0$, $X$ is contained in the normalizer $U$ in $G$ of the identity connected component $B_0$ of $B$. The complex coset space $G/X$, which coincides with $\overline{G}/\overline{U}$, is a Kaehler $C$-space. By Theorem 1, the factor group $X/B_0$ is a connected reductive complex Lie group. The fibre $\overline{U}/\overline{B}$ of the holomorphic fibre bundle $(\overline{G}/\overline{B}, \bar{p}, \overline{G}/\overline{U})$ is equal to $X/B$; indeed, we have a holomorphic homeomorphism of $X/B$ onto

$\overline{U}/\overline{B}$, which is induced from the injection of $X$ into $\overline{U}$. Moreover, we see that the homomorphism of $X/B_0$ into $\overline{U}/\overline{B}_0$ induced from the injection of $X$ into $\overline{U}$ is locally isomorphic and onto. This follows from the fact that the dimensions of $X/B_0$ and $\overline{U}/\overline{B}_0$ are equal to those of $X/B$ and $\overline{U}/\overline{B}$, respectively. Thus, $\overline{U}/\overline{B}_0$ is a reductive complex Lie group.

Next, we shall show that the dimension of the center of $\overline{G}$ is larger than or equal to that of the center of $\overline{U}/\overline{B}_0$. We regard $\overline{U}/\overline{B}_0$ as the structure group of the fibre bundle $(\overline{G}/\overline{B}, \bar{p}, \overline{G}/\overline{U})$. Then, the associated principal bundle is $(\overline{G}/\overline{B}_0, \bar{q}, \overline{G}/\overline{U})$, where $\bar{q}$ denotes the canonical projection from $\overline{G}/\overline{B}_0$ onto $\overline{G}/\overline{U}$. Let $\tilde{u}$ be a coset in $\overline{U}/\overline{B}_0$, and let $u$ be a representative of the coset $\tilde{u}$. The holomorphic homeomorphism defined by $g\cdot\overline{B}_0 \rightarrow gu\cdot\overline{B}_0$, $g\in\overline{G}$, is determined by the coset $\tilde{u}$, and is the right translation of the principal bundle $\overline{G}/\overline{B}_0$ corresponding to the element $\tilde{u}$ of the structure group. Evidently, the right translation commutes with the mapping $g\cdot\overline{B}_0\rightarrow xg\cdot\overline{B}_0$, $g\in\overline{G}$, assigned to an element $x$ in $\overline{G}$. We denote by $Z$ the identity connected component of the center of $\overline{U}/\overline{B}_0$, and by $n$ the complex dimension of $Z$. The complex Lie group $Z$, being a Lie subgroup in the structure group, acts on $\overline{G}/\overline{B}_0$ as a holomorphic transformation group. Let $X_1, \cdots, X_n$ be linearly independent holomorphic vector fields induced by one-parameter subgroups in $Z$. Then, at each point, they are linearly independent and form a base of the complex tangent space of the orbit of $Z$ through the point. Moreover, each $X_i$ is invariant by $\overline{G}$.

Denoting by $s$ the canonical projection of $\overline{G}/\overline{B}_0$ onto $\overline{G}/\overline{B}$, we obtain a holomorphic principal fibre bundle $(\overline{G}/\overline{B}_0, s, \overline{G}/\overline{B})$ whose structure group is the discrete subgroup $\overline{B}/\overline{B}_0$ in $\overline{U}/\overline{B}_0$. Since $Z$ is in the center of $\overline{U}/\overline{B}_0$, each of the holomorphic vector fields $X_1, \cdots, X_n$ is invariant by the action of the structure group $\overline{B}/\overline{B}_0$, and hence they are projectable. Let $Y_1, \cdots, Y_n$ be their image by the projection $s$. As $s$ is a local homeomorphism, $Y_1, \cdots, Y_n$ are linearly independent at each point. It is also obvious that each $Y_i$ is invariant by $\overline{G}$. The complex manifold $\overline{G}/\overline{B}$ being compact, the holomorphic vector fields $Y_1, \cdots, Y_n$ generate a connected complex Lie subgroup $\overline{Z}$ of dimension $n$ in $A_0(M)$. As $A_0(M)=\overline{G}$, $\overline{Z}$ is in the center of $\overline{G}$ and evidently in the radical $\overline{R}$ of $\overline{G}$.

In order to complete the proof, it suffices to show that the radical $\overline{R}$ of $\overline{G}$ is contained in the center of $\overline{G}$. Since $\overline{G}/\overline{U}$ is a Kaehler $C$-space, $\overline{R}$ is contained in $\overline{U}$, and so is $\overline{Z}$. First, we shall see that the image of $\overline{Z}\cdot\overline{B}$ under the canonical homomorphism $\tau$: $\overline{U}\rightarrow\overline{U}/\overline{B}_0$ contains $Z$. For this purpose, we recall how the group $\overline{Z}$ is constructed. Take an element $z$ in $\tau^{-1}(Z)$. To the right translation $f_{\tau(z)}: g\cdot\overline{B}_0 \to gz\cdot\overline{B}_0$, $g\in\overline{G}$, of the principal bundle $\overline{G}/\overline{B}_0$, there corresponds a holomorphic homeomorphism $g_{\tau(z)}$ of $\overline{G}/\overline{B}$ onto itself, such that $s\cdot f_{\tau(z)} = g_{\tau(z)}\cdot s$. Hence, $g_{\tau(z)}$ is the mapping $g\cdot\overline{B}\to gz\cdot\overline{B}$, $g\in\overline{G}$. On the other hand, $g_{\tau(z)}$ is realized by a mapping $g\cdot\overline{B}\to z'g\cdot\overline{B}$, $g\in\overline{G}$, for a certain element $z'$ in $\overline{Z}$. Therefore, we have $z\in z'\cdot\overline{B}$, and $Z\subset\tau(\overline{Z}\cdot\overline{B})$. Since the image of $\overline{R}$ under $\tau$ is in $Z$, we have $\overline{R}\subset\overline{Z}\cdot\overline{B}$. It follows that the orbit of $\overline{R}$ through a point is equal to the orbit of $\overline{Z}$; in fact, $\overline{R}\cdot g\cdot\overline{B} = g\cdot\overline{R}\cdot\overline{B} = g\cdot\overline{Z}\cdot\overline{B}$ for any $g\in\overline{G}$. Let $Y$ be a holomorphic vector field induced by a one-parameter subgroup in $\overline{R}$. Then, $Y$ is expressed as a linear combination of $Y_1, \cdots, Y_n$ whose coefficients are holomorphic functions on $\overline{G}/\overline{B}$. Since $\overline{G}/\overline{B}$ is compact, all the coefficients must be constant. Thus, we have seen that the radical $\overline{R}$ coincides with the central subgroup $\overline{Z}$, and accordingly the dimension of the center of $\overline{U}/\overline{B}_0$ is equal to that of the center in $\overline{G}$. From these facts, the assertion of Theorem 2 follows immediately.

REMARK. As an immediate implication of the above theorem, we see that if $M$ is a Kaehler $C$-space, then $A_0(M)$ is semi-simple. This is a corollary of a theorem obtained by Matsushima (see Nagoya Math. J. 11). Indeed, $M$ is a complex coset space of a connected complex semi-simple Lie group $G$ by a closed connected complex Lie subgroup $B$, whose normalizer coincides with itself [3, (5.2)]. We may assume that $G$ is a subgroup of $A_0(M)$. From what we have shown in the above proof, it follows that $\overline{U}/\overline{B}_0$ reduces to the identity and accordingly so does the identity connected component of the center of the complex reductive Lie group $A_0(M)$. Thus, $A_0(M)$ is semi-simple.

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WASHINGTON UNIVERSITY