

MONOTONICITY OF THE DIFFERENCES OF ZEROS OF BESSEL FUNCTIONS AS A FUNCTION OF ORDER¹

LEE LORCH AND PETER SZEGO²

1. **Introduction.** Throughout this note, $c_{\nu m}$ and $\gamma_{\nu k}$ denote, respectively, the m th and k th positive zeros of any pair (distinct or not) of real Bessel (cylinder)³ functions of order ν , arranged so that $c_{\nu m} > \gamma_{\nu k}$, where m and k (the respective ranks) are fixed positive integers. Each such zero increases with ν [6, p. 508].

In particular, we may take the Bessel functions involved to be identical and put $m = k + l$, where l is any positive integer. When $l = 1$, this specializes to the familiar differences which are defined in the usual way as

$$(1) \quad \Delta c_{\nu k} = c_{\nu, k+1} - c_{\nu k}; \quad \Delta^{n+1} c_{\nu k} = \Delta(\Delta^n c_{\nu k}), \quad n = 1, 2, \dots$$

The differences (1) have been investigated for all n , varying k and fixed order ν , where $\nu > \frac{1}{2}$, in [2; 3]. Here, on the contrary, our concern is with certain differences of zeros, each of fixed rank, as the order ν of the Bessel function varies.

In [2, §3] we showed, *i.a.*, that

$$(2) \quad (-1)^{n-1} \Delta^n c_{\nu k} > 0, \quad n, k = 1, 2, \dots, \text{ for } \nu > \frac{1}{2}.$$

Here our principal purpose is to prove that (i) *the positive quantity* $(-1)^{n-1} \Delta^n c_{\nu k}$ *increases with* ν *for each fixed* k , $n = 1, 2, \dots$, *when* $\nu > \frac{1}{2}$, *and that* (ii) *the difference* $(c_{\nu m} - \gamma_{\nu k})$ *increases with* ν *when* $\nu \geq 0$.⁴

For $n = 1$, (ii) is a two-fold generalization of (i): the difference considered is defined more generally, and the order ν covers a wider domain.

The method of proof of (ii) yields some information also for the case $\nu \leq -\frac{1}{2}$, showing that the difference $(c_{\nu m} - \gamma_{\nu k})$ *decreases* "in general" in this instance. However, here the rank of a zero can change because

Presented to the Society, January 24, 1963; received by the editors May 29, 1962 and, in revised form, October 18, 1962.

¹ The preparation of this note was facilitated by a grant from the General Research Funds of the University of Alberta.

² The second named author is now at the Ampex Corporation, Redwood City, California.

³ By a Bessel function, we mean any real solution, $\mathcal{C}_\nu(x) = J_\nu(x) \cos \theta - Y_\nu(x) \sin \theta$, of the Bessel differential equation, where the constant θ is independent of ν .

⁴ These results, like (2), may be helpful in checking tables of zeros of Bessel functions.

of the possible appearance or disappearance of zeros as ν varies through a certain discrete set of negative values [6, p. 509]. This creates obvious notational difficulties. To avoid them, we introduce the convention that $c_{\nu m}$, $\gamma_{\nu k}$ are to be understood (when ν is negative) as given functions of ν (m, k fixed) and that these functions are not to be altered even if a change in rank occurs. To emphasize this, we formulate the relevant results in terms of the derivatives with respect to ν of these functions.

2. Formal statement of results.

THEOREM 1. For $\nu > \frac{1}{2}$, the quantity $(-1)^{n-1} \Delta^n c_{\nu k}$ (which is positive [2, §3]) increases with ν for each pair of fixed positive integers k , $n = 1, 2, \dots$.

THEOREM 2. For each fixed pair of positive integers m, k , the difference $(c_{\nu m} - \gamma_{\nu k})$ increases with ν for $\nu \geq 0$ and

$$(3) \quad D_\nu(c_{\nu m} - \gamma_{\nu k}) < 0 \quad \text{for } \nu \leq -\frac{1}{2},$$

where D_ν denotes differentiation with respect to ν .

COROLLARY 1. For each fixed pair of positive integers k and l , the difference $(c_{\nu, k+l} - c_{\nu k})$ increases with ν for $\nu \geq 0$, and

$$(4)^5 \quad D_\nu(c_{\nu, k+l} - c_{\nu k}) < 0 \quad \text{for } \nu \leq -\frac{1}{2}.$$

COROLLARY 2. If $c_{\nu k}$ and $\gamma_{\nu k}$ are the respective k th positive zeros of linearly independent Bessel functions of order ν , and if $c_{\nu 1} > \gamma_{\nu 1}$, then the difference $(c_{\nu k} - \gamma_{\nu k})$ increases with ν for $\nu \geq 0$, and

$$(5) \quad D_\nu(c_{\nu k} - \gamma_{\nu k}) < 0 \quad \text{for } \nu \leq -\frac{1}{2},$$

for each fixed $k = 1, 2, \dots$, so that, in particular, the difference⁶ $(j_{\nu k} - y_{\nu k})$ increases with ν for $\nu \geq 0$, and

$$(6) \quad D_\nu(j_{\nu k} - y_{\nu k}) < 0 \quad \text{for } \nu \leq -\frac{1}{2},$$

for each fixed $k = 1, 2, \dots$.

Corollary 1 is an obvious consequence of Theorem 2. To derive Corollary 2 from that theorem, it is sufficient to note that the zeros $c_{\nu k}$, $\gamma_{\nu k}$ are interlaced [6, p. 481] and that $j_{\nu 1} > y_{\nu 1}$ [6, p. 487 (10)].

⁵ Formula (4), with $l = 1$, is an analogue for Bessel functions of results obtained by P. Turán [5, p. 115 (4.4), (5.1)] for the zeros of Legendre polynomials, and extended by G. Szegő and P. Turán [4, Theorems II and III] to ultraspherical polynomials of parameter α , $0 < \alpha < 1$.

⁶ As usual, $j_{\nu k}$ and $y_{\nu k}$ denote the respective k th positive zeros of $J_\nu(x)$ and $Y_\nu(x)$.

3. A lemma. The following result is useful in the proofs of both theorems.

LEMMA. *Let*

$$(7) \quad g(x) = 2x \int_0^{\infty} K_0(2x \sinh T) e^{-2\nu T} dT, \quad x > 0,$$

where K_0 is the Bessel function of second kind, imaginary argument and zeroth order. Then (for all ν)

$$(8) \quad g'(x) = 2 \int_0^{\infty} (\tanh T)(\tanh T + 2\nu) K_0(2x \sinh T) e^{-2\nu T} dT,$$

and

$$(9) \quad (-1)^{n-1} g^{(n)}(x) \left\{ \begin{array}{l} > 0 \text{ for } \nu \geq 0 \\ < 0 \text{ for } \nu \leq -\frac{1}{2} \end{array} \right\}, \quad n = 1, 2, \dots$$

PROOF. From the definition (7), we have

$$\begin{aligned} \frac{1}{2} g'(x) &= \int_0^{\infty} \{ K_0(2x \sinh T) + (2x \sinh T) K_0'(2x \sinh T) \} e^{-2\nu T} dT \\ &= \int_0^{\infty} \{ (\operatorname{sech}^2 T + \tanh^2 T) K_0(2x \sinh T) \\ &\quad + (2x \sinh T) K_0'(2x \sinh T) \} e^{-2\nu T} dT \\ &= \int_0^{\infty} \{ (\operatorname{sech}^2 T) K_0(2x \sinh T) + (2x \sinh T) K_0'(2x \sinh T) \} e^{-2\nu T} dT \\ &\quad + \int_0^{\infty} (\tanh^2 T) K_0(2x \sinh T) e^{-2\nu T} dT. \end{aligned}$$

Integrating the next to the last integral by parts, it follows that

$$\begin{aligned} \frac{1}{2} g'(x) &= (\tanh T) K_0(2x \sinh T) e^{-2\nu T} \Big|_{T=0}^{T=\infty} \\ &\quad + 2\nu \int_0^{\infty} (\tanh T) K_0(2x \sinh T) e^{-2\nu T} dT \\ &\quad + \int_0^{\infty} (\tanh^2 T) K_0(2x \sinh T) e^{-2\nu T} dT \\ &= \int_0^{\infty} (\tanh T)(\tanh T + 2\nu) K_0(2x \sinh T) e^{-2\nu T} dT, \end{aligned}$$

where the expression $(\tanh T)K_0(2x \sinh T)e^{-2\nu T}$ vanishes for $T=0$ and $T=\infty$, because [6, p. 80(14), p. 202(1)], $K_0(v) = O(\log v)$ as $v \rightarrow 0+$, and $K_0(v) = O(e^{-v})$ as $v \rightarrow \infty$.

This proves (8). In turn, (9) follows, on noting that [6, p. 446, footnote]

$$K_0(x) = \int_0^\infty e^{-x \cosh T} dT.$$

4. Proof of Theorem 1. From [6, p. 508(3)] we see that $D_\nu\{c_{\nu k}\} = g(c_{\nu k})$, where g is the function defined by (7). The difference operator being linear, we have

$$D_\nu\{\Delta^n c_{\nu k}\} = \Delta^n\{g(c_{\nu k})\}.$$

Now, employing [2, Lemma 2.3], we transform the Bessel differential equation, written in the form

$$(10) \quad y''(x) + \left(1 - \frac{\nu^2 - \frac{1}{4}}{x^2}\right)y(x) = 0,$$

into the differential equation

$$(11) \quad u''(t) + u(t) = 0,$$

so that the positive zeros of the solutions of (10), $c_{\nu k}$, are related to the positive zeros, t_k , of $u(t)$ by the equation $c_{\nu k} = c_\nu(t_k)$.⁷ Clearly, $\Delta t_k = \pi$ for all k .

Therefore,

$$(12) \quad D_\nu\{\Delta^n c_{\nu k}\} = \Delta^n\{g[c_\nu(t_k)]\}.$$

From the mean-value theorem for higher derivatives and differences [1, p. 73(1)], the right member of (12) can be written as $\pi^n D_i^n\{g[c_\nu(\xi)]\}$ for a suitable ξ . Hence,

$$(13) \quad \operatorname{sgn} D_\nu\{\Delta^n c_{\nu k}\} = \operatorname{sgn} D_i^n\{g[c_\nu(t)]\},$$

for $t = \xi$.

We show next that the right member of (13) is $(-1)^{n-1}$, even without the specialization $t = \xi$, and this will complete the proof of the theorem.

To this end, we recall from [2, §3 and Lemma 2.2 ($\sigma = 1$)] that

$$(14) \quad (-1)^{n-1} c_\nu^{(n)}(t) > 0, \quad n = 1, 2, \dots, \text{ for } \nu > \frac{1}{2}.$$

⁷ The function $c_\nu(t)$ is the function $x(t)$ of [2], where $c'_\nu(t) = x'(t) = p(x)$ as used in [2, §3].

Now, the chain rule for higher derivatives [1, §81, pp. 90-91] allows us to write

$$(15) \quad D_t^n \{g[c_r(t)]\} = \sum_{k=1}^n \left\{ \sum p_r [c_r'(t)]^{\alpha_1} \cdots [c_r^{(n)}(t)]^{\alpha_n} \right\} g^{(k)}(t),$$

where $p_r > 0$ (all r) and the summation inside the braces is taken over all non-negative integers $\alpha_1, \dots, \alpha_n$ such that

$$(16) \quad \alpha_1 + \cdots + \alpha_n = k, \quad \alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n.$$

Finally, we show that the sign of $g^{(k)}(t)$ times each term of the sum inside the braces in (15) is $(-1)^{n-1}$. Since $p_r > 0$, this sign is the same as

$$\text{sgn} \{ [c_r'(t)]^{\alpha_1} \cdots [c_r^{(n)}(t)]^{\alpha_n} g^{(k)}(t) \}$$

which, following (14), (16) and (9), is

$$\begin{aligned} & (1)^{\alpha_1} (-1)^{\alpha_2} (1)^{\alpha_3} \cdots [(-1)^{n-1}]^{\alpha_n} (-1)^{k-1} \\ &= (-1)^{2\alpha_1} (-1)^{3\alpha_2} (-1)^{4\alpha_3} \cdots (-1)^{(n+1)\alpha_n} (-1)^{k-1} \\ &= (-1)^{2\alpha_1 + 3\alpha_2 + 4\alpha_3 + \cdots + (n+1)\alpha_n + k-1} \\ &= (-1)^{\alpha_1 + 2\alpha_2 + 3\alpha_3 + \cdots + n\alpha_n + 2k-1} = (-1)^{n-1}. \end{aligned}$$

Thus, $g^{(k)}(t)$ times each term in the inner sum in (15) has sign $(-1)^{n-1}$, whence, a fortiori, so has $D_t^n \{g[c_r(t)]\}$ and the theorem is proved.

REMARK. Conceivably, Theorem 1 may be valid for a range of ν greater than $\nu > \frac{1}{2}$ for which we have established it, since some "balancing" in (15) may still leave that expression of appropriate sign without each term individually, as here, being of that sign.

5. Proof of Theorem 2. The restriction to (strictly) positive zeros assures the existence of $D_r(c_{\nu m} - \gamma_{\nu k})$, since $c_{\nu m}, \gamma_{\nu k}$ are analytic functions of ν wherever they are not zero [6, p. 509].

From [6, p. 508(3)] we have

$$(17) \quad D_r(c_{\nu m} - \gamma_{\nu k}) = g(c_{\nu m}) - g(\gamma_{\nu k}),$$

where g is defined by (7).

Since $c_{\nu m} > \gamma_{\nu k} > 0$, formula (8) shows that (17) is positive for $\nu \geq 0$ and negative for $\nu \leq -\frac{1}{2}$.

This proves the theorem and, with it, the corollaries.

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UNIVERSITY OF ALBERTA, EDMONTON AND
UNIVERSITY OF SANTA CLARA

COMPLETELY WELL-POSED PROBLEMS FOR NONLINEAR DIFFERENTIAL EQUATIONS

M. D. GEORGE¹

1. Well-posed and completely well-posed problems for linear partial differential equations have been discussed by Hormander [2] and more recently and more generally by Browder [1]. Roughly speaking, if L is a differential operator in a Banach space X , the problem of finding a solution of $Lu=f$, $f \in X$, is said to be (*completely*) *well-posed* if the range of L is X and if in addition L^{-1} exists and is (*completely*) continuous. In both papers, sufficient conditions are given for the existence of well-posed and completely well-posed problems for formal differential operators.

In this paper we are interested in the effect on a completely well-posed problem of a nonlinear perturbation of the operator L . In particular, we will show (Theorem 3) that under certain conditions a completely well-posed problem for a differential operator L remains completely well-posed for $L+A$, where A is a nonlinear transformation in X . Combining this result with theorems in [1], conditions guaranteeing the existence of completely well-posed problems for perturbed differential operators can be derived. One such result is given in Theorem 4 for the case $X=L^2$.

2. Let X be a Banach space, T a transformation with domain $D(T) \subset X$ and range $R(T) \subset X$. The transformations here are not assumed to be linear unless it is so stated.

Received by the editors November 5, 1962.

¹ This research was supported by the Air Force Office of Scientific Research.