

A CHARACTERIZATION OF THE MORSE MINIMAL SET

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1. Introduction. In order to prove the existence of "recurrent" non-periodic geodesics on certain surfaces of negative curvature, Marston Morse, in 1921, (cf. [1]), published an example of an unending indexed sequence of two symbols which is not periodic but almost periodic in the sense that every block which appears in the sequence appears infinitely often, to right and left, with bounded gaps.

In 1938 (cf. [2; 3]), Morse showed that his sequence had the interesting property that it did not contain any block of the form Bb , where B is a block, and b is the initial element of B .

Actually, the Morse sequence has been rediscovered independently at various times, in particular, by Arshon in 1937 (cf. [4]), and he showed that it contained no block BBB . Sequences with this property have useful applications in group-theoretic problems, particularly the Burnside problem.

The particular sequence constructed by Morse is not the only sequence of two symbols with the property of not containing a block of the form Bb , where b is the initial element of B . Any sequence, each of whose blocks appears in the Morse sequence, will have the same property. An uncountable set of such sequences exists and a specific construction of this set has been given by Kakutani (cf. [5]). Actually, these are precisely the sequences (points) of the Morse minimal set (cf. [5]).

We consider here the problem of determining the set of all sequences of two symbols with the property of not containing a block of the form Bb when b is the initial element of B and show that this set is precisely the collection of sequences constituting the Morse minimal set. Thus any such sequence contains all the blocks and only the blocks which appear in the Morse sequence. In view of this result, rediscovery of the Morse sequence cannot be regarded as surprising.

2. The sequence space S , the shift transformation, and minimal sets. Let S denote the set of all unending indexed sequences of 0's and 1's. Each member of S is a mapping of the set I of integers into the set $\{0, 1\}$. If $x \in S$ and $i \in I$ we will denote $x(i)$ by x_i .

Let $x, y \in S$. We define $d(x, y) = 0$ if $x = y$ and $d(x, y) = (1 + k)^{-1}$ if

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$x \neq y$, where k is the least non-negative integer such that either $x_{-k} \neq y_{-k}$ or $x_k \neq y_k$. This defines a metric in S and with this metric, S is a compact, totally disconnected, dense-in-itself set which is homeomorphic to the Cantor discontinuum.

Let $\sigma: S \rightarrow S$ be defined by $\sigma(x) = y$, $x \in S$, provided $y_{i+1} = x_i$, $i \in I$. This transformation is called the *shift transformation* and is a homeomorphism of S onto S .

A subset of S is a *minimal set* provided it is nonvacuous, closed, invariant under σ and contains no proper subset with the same properties.

An *n-block* or *block of length n* is a mapping from n successive integers into the set $\{0, 1\}$. The blocks $x_{i+1} \cdots x_{i+p}$ and $y_{j+1} \cdots y_{j+q}$ are identical if and only if $p = q$ and $x_{i+k} = y_{j+k}$, $k = 1, 2, \dots, p$. The block $B = b_1 b_2 \cdots b_n$ appears in the block $x_{i+1} x_{i+2} \cdots x_{i+p}$ if and only if $p \geq n$ and there exists k , $1 \leq k \leq p - n + 1$, such that $x_{i+k} x_{i+k+1} \cdots x_{i+k+n-1} = b_1 b_2 \cdots b_n$.

3. The Morse sequence and minimal set. The Morse sequence can be defined as follows (cf. [5]).

Let $B = b_1 b_2 \cdots b_n$ be an n -block. The block $\bar{B} = \bar{b}_1 \bar{b}_2 \cdots \bar{b}_n$ is defined by setting $\bar{b}_i = 0$ if $b_i = 1$ and $\bar{b}_i = 1$ if $b_i = 0$, $i = 1, 2, \dots, n$.

We define a sequence of blocks A_1, A_2, \dots inductively, by setting $A_1 = 0$ and $A_{n+1} = A_n \bar{A}_n$, $n \geq 1$. Then A_n is a 2^{n-1} -block.

Let $\mu \in S$ be defined by

$$\begin{aligned} \mu_0 \mu_1 \cdots \mu_{2^{n-1}-1} &= A_n, & n &\geq 1, \\ \mu_{-i} &= \mu_{i-1}, & i &\geq 1. \end{aligned}$$

It will be useful to observe also that all of the blocks $A_n A_n, A_n \bar{A}_n, \bar{A}_n A_n, \bar{A}_n \bar{A}_n$ appear in μ . This follows from the fact that

$$A_{n+3} = A_n \bar{A}_n \bar{A}_n A_n \bar{A}_n A_n \bar{A}_n.$$

The sequence μ is the *Morse sequence*. It is almost periodic, but not periodic (cf. [5]). It has the property that it contains no block of the form $B \bar{B} b_1 = b_1 b_2 \cdots b_n b_1 b_2 \cdots b_n b_1$ (cf. [2]).

Let M be the orbit-closure of μ under σ . The set M is minimal and is the *Morse minimal set*. If $x \in M$, then x is a limit point of the set $\{\sigma^i(\mu) \mid i \in I\}$ and consequently any block in x must appear in μ . Since M is a minimal set, the orbit of x is dense in M and consequently any block appearing in μ also appears in x . It follows that x has the property that it contains no block of the form $B \bar{B} b_1$, where b_1 is the first element of B .

4. Characterization of the sequences in the Morse minimal set M . Let $x \in S$. Then x has *property P* provided x does not contain a block

of the form $BBb_1 = b_1 \cdots b_m b_1 \cdots b_m b_1$. The set of all members of S with property P will be denoted by \mathcal{P} . We have observed that $M \subset \mathcal{P}$. We now show that $\mathcal{P} \subset M$.

LEMMA 1. *Let $x \in \mathcal{P}$. Then both of the blocks 00 and 11 appear in x .*

PROOF. Suppose 00 does not appear in x . The 1-block 0 must appear in x . For otherwise $x_i = 1$, $i \in I$, and the block 111 appears in x , contrary to hypothesis. Thus there exists $j \in I$ such that $x_j = 0$.

Since 00 does not appear in x , we have $x_{j-1}x_jx_{j+1} = 101$. Now $x_{j+2}x_{j+3}$ cannot be 00, since 00 does not appear; $x_{j+2}x_{j+3}$ cannot be 11, for then $x_{j+1}x_{j+2}x_{j+3} = 111$ and x does not have property P; $x_{j+2}x_{j+3}$ cannot be 01, for then $x_{j-1}x_jx_{j+1}x_{j+2}x_{j+3} = 10101$ and x does not have property P. Thus $x_{j+2}x_{j+3} = 10$. Similar argument shows that $x_{j-3}x_{j-2} = 01$. But then $x_{j-3}x_{j-2} \cdots x_{j+3} = 0110110$ and x does not have property P. Thus 00 must appear in x .

Suppose 11 does not appear in x . Let $\bar{x} \in S$ be defined by $\bar{x}_i = 0$ if $x_i = 1$ and $\bar{x}_i = 1$ if $x_i = 0$. Then 00 does not appear in \bar{x} and \bar{x} has property P, contrary to the first part of the proof. We conclude that 11 must appear in x .

LEMMA 2. *Let $x \in \mathcal{P}$. Then there exists $k \in I$ such that $x_{k+2n}x_{k+2n+1}$ is either 01 or 10, $n \in I$.*

PROOF. We assume $x \in S$ and x has property P. According to Lemma 1, 00 appears in x and thus there exists an integer k such that $x_{k-1}x_k = 00$. Since x has property P it follows that $x_{k-2} = x_{k+1} = 1$ and the conclusion of the lemma is valid for $n = -1$ and $n = 0$.

Proceeding by induction, we assume that $x_{k+2n}x_{k+2n+1}$ is either 01 or 10 for $n = -1, 0, 1, \dots, p$, and prove that $x_{k+2p+2}x_{k+2p+3}$ is either 01 or 10. For suppose $B = x_{k+2p-2}x_{k+2p-1}x_{k+2p}x_{k+2p+1} = 0101$. If $x_{k+2p+2}x_{k+2p+3} = 00$ then x contains 01010 and does not have property P. If $x_{k+2p+2}x_{k+2p+3} = 11$ then x contains the block 111 and does not have property P. Thus $x_{k+2p+2}x_{k+2p+3}$ is either 01 or 10.

Now suppose $B = 1001$. If $x_{k+2p+2}x_{k+2p+3} = 00$, then x_{k+2p+4} must be 1 and x contains the block 1001001 and x does not have property P. Thus $x_{k+2p+2}x_{k+2p+3}$ is not 00. If $x_{k+2p+2}x_{k+2p+3} = 11$, then x contains the block 111 and x does not have property P. Thus $x_{k+2p+2}x_{k+2p+3}$ is either 01 or 10.

We assume $B = 0110$. If $x_{k+2p+2}x_{k+2p+3} = 00$, then x contains the block 000, contrary to hypothesis. If $x_{k+2p+2}x_{k+2p+3} = 11$, then x_{k+2p+4} must be 0 and x contains the block 0110110, contrary to hypothesis. It follows that $x_{k+2p+2}x_{k+2p+3}$ must be either 01 or 10.

Suppose $B = 1010$. Then $x_{k+2p+2} = 0$, for otherwise x contains 10101,

contrary to hypothesis. If $x_{k+2p+3}=0$, then x contains the block 000, contrary to hypothesis. Thus $x_{k+2p+2}x_{k+2p+3}=01$.

By induction we infer that $x_{k+2n}x_{k+2n+1}$ is either 01 or 10 for $n \geq -1$.

The proof that $x_{k+2n}x_{k+2n+1}$ is either 01 or 10 for $n \leq -1$ is similar.

The proof is completed.

NOTATION. Let $x \in S$, let $m \in I$ and let $n \in I^+$. Here I^+ stands for the set of all positive integers. Then $B(x, m, n)$ will denote the n -block $x_{m+1}x_{m+2} \cdots x_{m+n}$.

LEMMA 3. Let $x \in \mathcal{O}$ and let there exist $k \in I, p \in I^+$ and p -blocks $A(0), A(1)$ such that for each $m \in I, B(x, k+mp, p)$ is either $A(0)$ or $A(1)$. Let $y_m = 0$ if $B(x, k+mp, p) = A(0)$ and let $y_m = 1$ if $B(x, k+mp, p) = A(1)$. Then $y \in \mathcal{O}$.

PROOF. Suppose $y \notin \mathcal{O}$ and thus a block $DDd = d_1d_2 \cdots d_qd_1d_2 \cdots d_qd_1$ appears in y . But then the block $A(d_1)A(d_2) \cdots A(d_q)A(d_1) \cdots A(d_q)A(d_1)$ appears in x , contrary to the hypothesis that $x \in \mathcal{O}$.

LEMMA 4. Let $x \in \mathcal{O}$ and let $n \in I^+$. Then there exists $k \in I$ such that for each $m \in I, B(x, k+m2^{n-1}, 2^{n-1})$ is either A_n or \bar{A}_n .

PROOF. Since $A_1=0$ and $\bar{A}_1=1$, the statement of the lemma is true for $n=1$. We assume the statement true for $n=p-1$ and prove that it is true for $n=p$. Thus there exists j such that

$$B(x, j + m2^{p-2}, 2^{p-2})$$

is either A_{p-1} or \bar{A}_{p-1} . Let y be defined by $y_m = 0$ if $B(x, j+m2^{p-2}, 2^{p-2}) = A_{p-1}, y_m = 1$ if $B(x, j+m2^{p-2}, 2^{p-2}) = \bar{A}_{p-1}$. According to Lemma 3, $y \in \mathcal{O}$. It follows from Lemma 2 that there exists $i \in I$ such that $y_{i+2q}y_{i+2q+1}$ is either 01 or 10, $q \in I$. But then either

$$B(x, j + (i + 2q)2^{p-2}, 2^{p-2})B(x, j + (i + 2q + 1)2^{p-2}, 2^{p-2}) = A_{p-1}\bar{A}_{p-1} = A_p$$

or

$$B(x, j + (i + 2q)2^{p-2}, 2^{p-2})B(x, j + (i + 2q + 1)2^{p-2}, 2^{p-2}) = A_{p-1}A_{p-1} = A_p.$$

Since

$$B(x, j + (i + 2q)2^{p-2}, 2^{p-2})B(x, j + (i + 2q + 1)2^{p-2}, 2^{p-2}) = B(x, j + i2^{p-2} + q2^{p-1}, 2^{p-1}),$$

if we choose $k=j+i2^{p-2}$, the statement of the lemma is true for $n=p$ and the proof by induction is completed.

THEOREM. Let $x \in \mathcal{O}$. Then $x \in M$.

PROOF. Let $x \in \mathcal{O}$. It is sufficient to prove that any block which appears in x appears in the Morse sequence μ . Let $B = x_{i+1}x_{i+2} \cdots x_{i+p}$. Choose $n \in I$ so that $2^{n-1} > p$. According to Lemma 4 there exists $k \in I$ such that for each $m \in I$, $B(x, k + m2^{n-1}, 2^{n-1})$ is either A_n or \bar{A}_n . Let m be the greatest integer such that $k + m2^{n-1} \leq i$. Then $k + (m+2)2^{n-1} > i+p$ and B is a subblock of

$$B(x, k + m2^{n-1}, 2^n) \\ = B(x, k + m2^{n-1}, 2^{n-1})B(x, k + (m+1)2^{n-1}, 2^{n-1}).$$

It follows from Lemma 4 that B is a subblock of at least one of the blocks $A_n A_n$, $A_n \bar{A}_n$, $\bar{A}_n A_n$, $\bar{A}_n \bar{A}_n$. Since all of these appear in the Morse sequence μ , the statement of the theorem is proved.

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