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COMPLETELY WELL-POSED PROBLEMS FOR NONLINEAR DIFFERENTIAL EQUATIONS

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1. Well-posed and completely well-posed problems for linear partial differential equations have been discussed by Hormander [2] and more recently and more generally by Browder [1]. Roughly speaking, if L is a differential operator in a Banach space X, the problem of finding a solution of $Lu=f, f\in X$, is said to be (*completely*) *well-posed* if the range of L is X and if in addition L^{-1} exists and is (completely) continuous. In both papers, sufficient conditions are given for the existence of well-posed and completely well-posed problems for formal differential operators.

In this paper we are interested in the effect on a completely well-posed problem of a nonlinear perturbation of the operator L. In particular, we will show (Theorem 3) that under certain conditions a completely well-posed problem for a differential operator L remains completely well-posed for $L+A$, where A is a nonlinear transformation in X. Combining this result with theorems in [1], conditions guaranteeing the existence of completely well-posed problems for perturbed differential operators can be derived. One such result is given in Theorem 4 for the case $X=L^2$.

2. Let X be a Banach space, T a transformation with domain $D(T)\subset X$ and range $R(T)\subset X$. The transformations here are not assumed to be linear unless it is so stated.

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DEFINITION. The transformation T is said to be *asymptotic to zero* if $D(T)=X$ and

$$\lim_{\|u\|\to\infty} \frac{\|Tu\|}{\|u\|} = 0.$$

This definition is due to Krasnoselskiĭ and the following theorem, referred to in [3], to Dubrovskiĭ, who used it in treating nonlinear integral equations. A proof of this theorem is included here for the convenience of the reader.

THEOREM 1. *If T is completely continuous and asymptotic to zero, then $R(I+T)=X$.*

PROOF. Let f be an arbitrary element of X. To prove that $u+Tu=f$ has a solution, it suffices to prove that the transformation S defined by $Su=f-Tu$ has a fixed point. Noting that S is completely continuous since T is, we need only show, using the Schauder theorem, that S maps some closed sphere of X into itself.

Let

$$B_r(f) = \{v \in X \mid \|v-f\| \leqq r\} \tag{1}$$

and suppose that for each integer $n>0$, the set $SB_n(f)$ contains an element Su_n not in $B_n(f)$. Then the sequence $\{u_n\}$ has the property that $\|u_n-f\| \leqq n$, while $\|Su_n-f\| = \|Tu_n\| > n$. Since T is completely continuous and the sequence $\{Tu_n\}$ is unbounded, $\{u_n\}$ is also unbounded. On the other hand $\|u_n\| \leqq \|f\| + n$, so that

$$\frac{\|Tu_n\|}{\|u_n\|} > \frac{n}{\|f\|+n}.$$

But this contradicts the assumption that T is asymptotic to zero, since for such an operator $\|Tu\|/\|u\|$ cannot be bounded away from zero as u ranges over an unbounded set. Consequently, S maps some $B_n(f)$ into itself, completing the proof.

The main result of this section is:

THEOREM 2. *Let L be a linear transformation, not necessarily bounded, with domain $D(L) \subset X$ and $R(L)=X$, and suppose L has a completely continuous inverse. Let A be bounded, continuous, and asymptotic to zero. Then $R(L+A)=X$.*

The proof will follow easily from Theorem 1 with the aid of the following lemma.

LEMMA. *If $D(A)=D(K)=X$, and K is linear and bounded while A*

is bounded and asymptotic to zero, then AK is asymptotic to zero.

PROOF. It will be convenient to introduce some additional notation. Since A is bounded, there is for each $r \geqq 0$ a number $s \geqq 0$ such that $AB_r(0) \subset B_s(0)$, using the notation introduced in (1). Denote by $M(r)$ the greatest lower bound of the set of such s. Also, since A is asymptotic to zero, there is a real non-negative function $P(\epsilon)$ defined for $\epsilon > 0$ such that $\|Av\| < \epsilon\|v\|$ whenever $\|v\| > P(\epsilon)$. Given $\epsilon > 0$, set $\rho = P(\epsilon/\|K\|)$ (if $K = 0$ the lemma is trivial) and let u be any element of X such that $\epsilon\|u\| > M(\rho)$. There are two cases to consider, according as $\|Ku\| > \rho$ or $\|Ku\| \leqq \rho$. In the first case,

$$\frac{\|AKu\|}{\|u\|} = \frac{\|AKu\|}{\|Ku\|}\,\frac{\|Ku\|}{\|u\|} \leqq \frac{\|AKu\|}{\|Ku\|}\|K\| < \frac{\epsilon}{\|K\|}\|K\| = \epsilon$$

while in the second case

$$\frac{\|AKu\|}{\|u\|} < \frac{M(\rho)}{M(\rho)/\epsilon} = \epsilon$$

so that in any event, $\|AKu\| < \epsilon\|u\|$ for $\|u\|$ sufficiently large.

PROOF OF THEOREM 2. Since $D(L + A) = D(L)$, $(L + A)L^{-1} = I + AL^{-1} = I + T$ is everywhere defined and from the lemma, T is asymptotic to zero. Furthermore, since A is continuous and L^{-1} completely continuous, T is completely continuous. Thus $R(I+T) = X$ by Theorem 1 and since $R(L) = X$ by hypothesis, it follows that $R(L+A) = R((I+T)L) = X$.

It should perhaps be noted here that Theorem 1 remains true under weaker hypotheses. For example, as is clear from the proof, one need only assume that for $\|u\|$ sufficiently large, $\|Tu\| \leqq c\|u\|$ for some $c < 1$. However, if A has only this weaker property, $T = AL^{-1}$ need not have, unless some restriction is made on $\|L^{-1}\|$.

3. Let K be any transformation in X. The problem of finding a solution $u \in D(K)$ of $Ku = f$, $f \in X$, is said to be (*completely*) *well-posed* if $R(K) = X$ and K has a (completely) continuous inverse. In this section we wish to consider the case in which K is a differential operator and X is a complex L^p space, $1 \leqq p < \infty$. The description which follows is admittedly brief; full details can be found in [1].

Let G be a bounded, open subset of Euclidean n-space, $n \geqq 1$. In now standard notation, we denote by

$$P = \sum_{|\alpha| \leq m} a_\alpha(x) D^\alpha$$

a linear differential operator of order m, with coefficients $a_\alpha(x)$ complex-valued functions on G, and by P', defined by

$$P'u = \sum_{|\alpha| \leq m} D^\alpha(\bar{a}_\alpha(x)u)$$

its formal adjoint. We first consider P and P' defined on $C_0^\infty(G)$, the infinitely-differentiable complex functions with compact support in G, and then close them as operators in $L^p(G)$ and $L^{p'}(G)$ respectively, where $p' = p/(p-1)$. These new operators will be denoted by P_0 and P_0^*. P_0 and P_1, the restricted adjoint of P_0^*, are called the minimal and maximal operators associated with the formal differential operator P. Let L be a closed linear operator with $P_0 \subseteq L \subseteq P_1$. If $R(L) = L^p(G)$ and L has a (completely) continuous inverse, then L is said to be a (*completely*) *solvable realization* of the pair (P_0, P_0^*) and the problem $Lu = f$ is then (completely) well-posed.

THEOREM 3. *Let $g(x, z)$ be a complex-valued function defined and uniformly continuous for $x \in G$, all complex z, such that*

$$|g(x, z)| \leqq c_1 |z|^a + c_2, \tag{2}$$

where c_1, c_2, and a are non-negative constants and $a < 1$. Denote by A the operator in $L^p(G)$ defined by $Au(x) = g(x, u(x))$. If L is a competely solvable realization of the pair (P_0, P_0^*), then the problem $(L+A)u = f$, $f \in L^p(G)$, is completely well-posed provided $L+A$ has a continuous inverse.

PROOF. The conditions imposed on $g(x, z)$ insure that A is a bounded and continuous operator defined on $L^p(G)$. (Actually, less will suffice to give the same result. References to papers giving results along this line can be found in [3].) That A is asymptotic to zero follows easily from (2) and hence $R(L+A) = L^p(G)$ from Theorem 2. Since $L+A$ has a continuous inverse by hypothesis, it remains only to show that in fact $(L+A)^{-1}$ is completely continuous. In view of the complete continuity of L^{-1}, it suffices to show that if $\{u_n\}$ is any sequence of elements in $D(L) = D(L+A)$ such that the set $\{(L+A)u_n\}$ is bounded, then the set $\{Lu_n\}$ is also bounded. Suppose then that $\|(L+A)u_n\| < M$ while $\{Lu_n\}$ is unbounded. Eliminating those n's for which $Lu_n = 0$ and setting $v_n = Lu_n$ for the remainder, we have $\|AL^{-1}v_n\| = \|Au_n\| \geqq \|Lu_n\| - M = \|v_n\| - M$, so that

$$\frac{\|AL^{-1}v_n\|}{\|v_n\|} \geqq 1 - \frac{M}{\|v_n\|}.$$

But, as in the proof of Theorem 1, this last inequality is impossible since AL^{-1} is asymptotic to zero. Thus $\{Lu_n\}$ is a bounded set and so $\{u_n\}$ has a convergent subsequence.

It is clear that the theorem could be immediately generalized to transformations of the form $L+KAM$, where A is as described in the theorem and K and M are linear and bounded. That KAM is asymptotic to zero follows easily from the lemma.

Finally, a word may be said about the existence of completely well-posed problems for the formal differential operator $P+A$. By combining these results with theorems of [1] and [2], a number of results can be obtained. One example will suffice to illustrate the idea.

THEOREM 4. *Let $p=2$ and let $Au=g(x, u(x))$, where $g(x, z)$ satisfies the conditions of Theorem 3. Suppose P_0 and P_0^* have completely continuous inverses on their respective ranges. Then there exists a completely solvable realization L of the pair (P_0, P_0^*) and a number $c>0$ such that if $|g(x, u)-g(x, v)| \leqq c|u-v|$ for all complex u, v and all x in G, then the problem $(L+A)u=f$, $f \in L^2(G)$, is completely well-posed.*

PROOF. The existence of L follows from Theorem 1.2 of [2]. (The corresponding theorem for reflexive Banach spaces can be found in [1].) An easy calculation shows that if $|g(x, u)-g(x, v)| \leqq c|u-v|$, where $c\|L^{-1}\|<1$, then for $u(x)$, $v(x) \in D(L)=D(L+A)$,

$$\|u - v\| \leqq \|L^{-1}\|(1 - c\|L^{-1}\|)^{-1}\|(L + A)u - (L + A)v\|.$$

Hence $L+A$ has a continuous inverse and the conclusion follows from Theorem 3.

Note that the constant c can be estimated from the construction of L^{-1} given in the proof of Theorem 1.2 in [2]. Hörmander also gives in [2] conditions which insure the complete continuity of P_0^{-1} and $(P_0^*)^{-1}$.

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