RELATIONS AMONG STIEFEL WHITNEY CLASSES

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The object of this paper is to consider the problems: What are the relations among the Whitney classes of all manifolds of dimension n? In other words, which $\alpha \in H^*(BO, Z_2)$ have the property that $\alpha(M^n)=0$ for all n-dimensional manifolds.

For a connected manifold M^n, we denote by I_M the kernel of the homomorphism $\tau^*: H^*(BO, Z_2) \to H^*(M^n, Z_2)$ induced by the classifying map for the tangent bundle of M^n.

If M and N are connected n-dimensional manifolds, and if we form the connected sum [2] of M and N, which will be denoted $M \# N$, then in dimensions between 0 and n, the cohomology of $M \# N$ can be decomposed into the direct sum of the cohomology of M and that of N. Further, this decomposition is compatible with the maps from $H^*(BO, Z_2)$. Thus, in dimensions between 0 and n, $I_{M\#N} = I_M \cap I_N$.

To decrease the ideal associated with a manifold, it is then reasonable to form the connected sum with another manifold. This process will be used to show:

THEOREM. *There is no relation of dimension less than or equal to the integral part of $n/2$ among the Whitney classes of all manifolds of dimension n.*

This result has also been obtained by an entirely different method by E. H. Brown, Jr. [1].

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PROOF OF THE THEOREM. Let $M_{2k} = \#_{(a_1, \cdots, a_r) \in \pi(k)} RP_{2a_1} \times \cdots \times RP_{2a_r}$ be the connected sum over all elements of the set of partitions of the integer k of the products of even dimensional real projective spaces.

(1) It suffices to show that for each k, $\tau^*: H^k(BO, Z_2) \to H^k(M_{2k}, Z_2)$ is a monomorphism.

Let $\alpha \in H^m(BO, Z_2)$ with $\alpha(M)=0$ for all manifolds M of dimension n, and suppose $m \leqq [n/2]$. If $n=2k$, let $M^n = M_{2k}$; if $n=2k+1$, let $M^n = M_{2k} \times S^1$. Then $m \leqq k$, and so $\tau_M^*(\alpha \cdot w_{k-m}) = \tau_M^*(\alpha) \cdot \tau_M^*(w_{k-m}) = 0$.

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If $n=2k$, this makes $\alpha\cdot w_{k-m}=0$, for τ^* is assumed monic. If $n=2k+1$, $w(M_{2k}\times S^1)=w(M_{2k})\otimes w(S^1)=w(M_{2k})\otimes 1$ in $H^*(M_{2k}\times S^1, Z_2)$, so $\tau_M^*(\alpha\cdot w_{k-m})=\tau^*(\alpha\cdot w_{k-m})\otimes 1$. Since τ^* is assumed monic, this also gives $\alpha\cdot w_{k-m}=0$. Since $H^*(BO, Z_2)$ is a polynomial algebra over Z_2, it has no divisors of zero and hence $\alpha=0$.

(2) Let $\overline{M}_{2k}$ denote the disjoint union $\bigcup_{(a_1,\cdots,a_r)\in\pi(k)} RP_{2a_1}\times\cdots\times RP_{2a_r}$. The cohomology of $\overline{M}_{2k}$ is the direct sum of the algebras $H^*(RP_{2a_1}\times\cdots\times RP_{2a_r}, Z_2)$ whose structure is well known. The Mayer-Vietoris sequence shows that $H^k(M_{2k}, Z_2)$ is isomorphic to $H^k(\overline{M}_{2k}, Z_2)$ under an isomorphism ϕ such that

$$\begin{array}{ccc} H^k(BO, Z_2) & \xrightarrow{\tau^*} & H^k(M_{2k}, Z_2) \\ & \searrow^{\bar\tau^*} & \downarrow \phi \\ & & H^k(\overline{M}_{2k}, Z_2) \end{array}$$

commutes.

Define a homomorphism $\Omega: H^k(BO, Z_2)\to H^{2k}(\overline{M}_{2k}, Z_2)$ by

$$\begin{array}{ccc} H^k(BO, Z_2) & \xrightarrow{\bar\tau^*} & H^k(\overline{M}_{2k}, Z_2) \\ Sq^k \downarrow & & \downarrow Sq^k \\ H^{2k}(BO, Z_2) & \xrightarrow{\bar\tau^*} & H^{2k}(\overline{M}_{2k}, Z_2). \end{array}$$

Since $\Omega=Sq^k\cdot\bar\tau^*=Sq^k\cdot\phi\cdot\tau^*$, it will suffice to show that Ω is an isomorphism.

(3) Consider the elements $w_i\in H^*(BO, Z_2)$ as the ith elementary symmetric functions σ_i of variables t_α. For each partition $(i_1,\cdots,i_r)$ in $\pi(k)$, we define $s_{(i_1,\cdots,i_r)}(\sigma_1,\cdots,\sigma_k)$ to be the symmetric function $\sum t_1^{i_1}\cdots t_r^{i_r}$. As is well known, the functions $s_\omega(\sigma_1,\cdots,\sigma_k)$ for $\omega\in\pi(k)$ form an additive basis for the symmetric functions in the t's which are homogeneous of degree k. Thus the elements $s_\omega(w_1,\cdots,w_k)$ for $\omega\in\pi(k)$ form an additive basis of $H^k(BO, Z_2)$. (See [3].)

For each $\omega=(i_1,\cdots,i_r)\in\pi(k)$, let $2\omega=(2i_1,\cdots,2i_r)\in\pi(2k)$, and let $RP_{2\omega}=RP_{2i_1}\times\cdots\times RP_{2i_r}$. Considering $H^{2k}(\overline{M}_{2k}, Z_2)$ as $\sum\oplus H^{2k}(RP_{2\omega}, Z_2)$, it will then suffice to show that the matrix

$$\left\|Sq^k\cdot\bar\tau^* s_\omega(RP_{2\tilde\omega})\right\|_{\omega,\tilde\omega\in\pi(k)}$$

over Z_2 is nonsingular, in order to show that Ω is an isomorphism.

(4) Since $Sq^k\bar\tau^*=\bar\tau^*Sq^k$, we consider the elements $Sq^k s_\omega\in H^{2k}(BO, Z_2)$. Since $s_\omega\in H^k(BO, Z_2)$,

$$\begin{aligned} Sq^k s_\omega &= Sq^k s_{(i_1,\cdots,i_r)} = (s_{(i_1,\cdots,i_r)})^2 = \left(\sum t_1^{i_1}\cdots t_r^{i_r}\right)^2 \\ &= \sum t_1^{2i_1}\cdots t_r^{2i_r} = s_{2\omega}. \end{aligned}$$

Thus the matrix $\|Sq^k\bar{\tau}^*s_\omega(RP_{2\tilde{\omega}})\|_{\omega,\ \in\pi(k)}$ can be considered as a submatrix of

$$\|\bar{\tau}^*s_\mu(RP_\lambda)\|_{\lambda,\mu\in\pi(2k)}.$$

Now the element

$$\begin{aligned}\bar{\tau}^*s_\mu(RP_\lambda) &= \bar{\tau}^*s_\mu(RP_{b_1}\times\cdots\times RP_{b_t}),\\ &= \sum_{\mu_1,\cdots,\mu_r=\mu}\bar{\tau}^*s_{\mu_1}(RP_{b_1})\otimes\cdots\otimes\tau^*s_{\mu_t}(RP_{b_t}),\end{aligned}$$

and so $\bar{\tau}^*s_\mu(RP_\lambda)=0$ unless μ refines λ. (See [3].) Putting a total order on the set $\pi(2k)$, compatible with the relation $\omega\geqq\tilde{\omega}$ if ω is a refinement of $\tilde{\omega}$, makes the matrix $\|\bar{\tau}^*s_\mu(RP_\lambda)\|$ a triangular matrix. Hence also $\|Sq^k\bar{\tau}^*s_\omega(RP_{2\tilde{\omega}})\|$ is triangular and has diagonal elements

$$\{\bar{\tau}^*s_{2\omega}(RP_{2\omega})\}.$$

If $\omega=(a_1,\cdots,a_r)$, this becomes

$$\bar{\tau}^*s_{2\omega}(RP_{2\omega}) = \bar{\tau}^*s_{2a_1}(RP_{2a_1})\otimes\cdots\otimes\bar{\tau}^*s_{2a_r}(RP_{2a_r}).$$

Since $w(RP_{2a_p})=(1+\alpha)^{2a_p+1}$, for $\alpha\in H^1(RP_{2a_p}, Z_2)$, $w_i(RP_{2a_p})$ can be considered as the ith elementary symmetric function in the $2a_p+1$ variables $\alpha,\cdots,\alpha$. Thus

$$\begin{aligned}\bar{\tau}^*s_{2a_p}(RP_{2a_p}) &= \sum(\alpha)^{2a_p},\\ &= (2a_p+1)\alpha^{2a_p},\\ &= \alpha^{2a_p},\end{aligned}$$

and so

$$\bar{\tau}^*s_{2\omega}(RP_{2\omega}) = \alpha_1^{2a_1}\otimes\cdots\otimes\alpha_r^{2a_r},$$

which when considered as a Whitney number is 1.

This means that the matrix $\|Sq^k\bar{\tau}^*s_\omega(RP_{2\tilde{\omega}})\|$ is triangular, with all diagonal entries having value 1, and hence this makes the matrix nonsingular, completing the proof of the theorem.

References

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