ON WRONSKIANS WHOSE ELEMENTS ARE ORTHOGONAL POLYNOMIALS[1]

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In a recent paper S. Karlin and G. Szegö [1], among many other topics, proved theorems concerning the sign and real zeros of the Wronskian

$$W(n, l; x) = \begin{vmatrix} Q_n(x) & Q_{n+1}(x) & \cdots & Q_{n+l-1}(x) \\ Q_n'(x) & Q_{n+1}'(x) & \cdots & Q_{n+l-1}'(x) \\ \vdots & \vdots & & \vdots \\ Q_n^{(l-1)}(x) & Q_{n+1}^{(l-1)}(x) & \cdots & Q_{n+l-1}^{(l-1)}(x) \end{vmatrix} \tag{1}$$

where $\{Q_r(x)\}$ $(r=0, 1, 2, \cdots)$ is a sequence of orthogonal polynomials associated with a certain measure. They proved [1, Theorem 1] if l is even, $W(n, l; x)$ keeps constant sign for all real x, i.e., the Wronskian (1), which is an $n.l$ degree polynomial, has only complex roots. In this note we want to show that if $\{Q_r(x)\}$ are orthogonal polynomials on a *finite* interval, we can obtain additional information on the location of *all* the zeros of (1).

THEOREM 1. *Let $q(x)$ be a positive continuous function on the finite interval $a \leqq x \leqq b$; and let $\{Q_r(x)\}$ $(r=0, 1, 2, \cdots)$ be a sequence of polynomials[2] which satisfy $\int_a^b q(x)Q_n(x)Q_m(x)dx=0$, $n\neq m$, then all the zeros of the Wronskian* (1) *are in $S=S[(a, b); \pi/l]$, where $S[(a, b); \Phi]$ denotes the region comprised of all points at which the interval $[a, b]$ subtends an angle of at least Φ.*

For the proof of Theorem 1 we need the following:

LEMMA I. *Let $\rho(x)$ be an mth degree polynomial; let $\alpha(x) \geqq 0$, $a \leqq x \leqq b$ be continuous and suppose*

$$\int_a^b \alpha(x)\rho(x)dx = 0. \tag{2}$$

Then $\rho(x)$ has at least one zero in $S[(a, b), \pi/m]$.

The above lemma is a special case of Marden's Mean-Value theo-

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[2] $Q_r(x)$ denotes a polynomial of the exact degree r.

rem [2, p. 87 (Theorem 24.3)]. The proof for $\alpha(x) \equiv 1$ can be found in [3].

With the aid of Lemma I we prove

LEMMA II. *Let* $\{Q_r(x)\}$ $(r=0, 1, 2, \cdots)$ *be as in Theorem* 1, *then for any complex numbers* a_ν *the polynomial*:

$$P(x) = \sum_{\nu=0}^{l-1} a_\nu Q_{n+\nu}(x) \tag{3}$$

has at least n *zeros in* $S[(a, b); \pi/l]$.

PROOF. Suppose that there are only $k \leqq n-1$ zeros $\zeta_1, \zeta_2, \cdots, \zeta_k$ of $P(x)$ in S, and the remaining zeros $\eta_{k+1}, \eta_{k+2}, \cdots, \eta_{n+l-1}$ are outside. Make use of Lemma I, letting

$$\begin{aligned}(4)\quad \alpha(x) = q(x)[(x - \zeta_1)(x - \bar{\zeta}_1)] \cdots [(x - \zeta_k)(x - \bar{\zeta}_k)] \\ \cdot [(x - \eta_{k+1})(x - \bar{\eta}_{k+1})] \cdots [(x - \eta_{n-1})(x - \bar{\eta}_{n-1})],\end{aligned}$$

$$(5)\quad \rho(x) = (x - \eta_n)(x - \eta_{n+1}) \cdots (x - \eta_{n+l-1}).$$

From the definition of $\alpha(x)$ in (4) it is clear that $\alpha(x) \geqq 0$ for $a \leqq x \leqq b$. The functions in (4), (5) also satisfy (2), for

$$\begin{aligned}\rho(x)\alpha(x) &= q(x)[(x - \zeta_1) \cdots (x - \zeta_k)(x - \eta_{k+1}) \cdots (x - \eta_{n+l-1})] \\ &\qquad \cdot [(x - \bar{\zeta}_1) \cdots (x - \bar{\eta}_{n-1})] \\ &= c\left[q(x) \sum_{\nu=0}^{l-1} a_\nu Q_{n+\nu}(x)\right] \pi_{n-1}(x) \qquad (c \neq 0)\end{aligned} \tag{6}$$

where

$$(7)\quad \pi_{n-1}(x) = (x - \bar{\zeta}_1)(x - \bar{\zeta}_2) \cdots (x - \bar{\zeta}_k)(x - \bar{\eta}_{k+1}) \cdots (x - \bar{\eta}_{n-1})$$

is an $n-1$ degree polynomial and we obtain:

$$\int_a^b \alpha(x)\rho(x)\, dx = c \sum_{\nu=0}^{l-1} a_\nu \int_a^b q(x) Q_{n+\nu}(x) \pi_{n-1}(x)\, dx = 0. \tag{8}$$

We conclude from Lemma I that $\rho(x)$ has at least one zero in $S[(a, b); \pi/l]$, which contradicts the definition of $\rho(x)$.

For the proof of Theorem 1 assume that there is at least one zero z_0 which is not in S, then the system of l linear equations in a_ν:

$$\sum_{\nu=0}^{l-1} a_\nu Q_{n+\nu}^{(m)}(z_0) = 0 \qquad (m = 0, 1, 2, \cdots, l-1) \tag{9}$$

has a nontrivial solution $(W(n, l; z_0) = 0)$. Consider the $n+l-1$ degree polynomial

$$P(x) = \sum_{\nu=0}^{l-1} a_\nu Q_{n+\nu}(x) \tag{10}$$

which by (9) has an l-fold zero at $x=z_0$. By our lemma the polynomial (10) has at least n zeros in S, i.e., the $n+l-1$ degree polynomial $P(x)$ has at least $n+l$ zeros, which is impossible. Theorem 1 is thus proved.

From Theorem 1 we obtain by elementary geometrical consideration the following

COROLLARY I. *Let* $\{Q_r(x)\}$ $(r=0, 1, 2, \cdots)$ *be as in Theorem* 1, *then* $W(n, l; x)$ *has no zeros in the sectors*: $\pi-\pi/l<\operatorname{Arg}(z-a)<\pi+\pi/l$ *and* $-\pi/l<\operatorname{Arg}(z-b)<\pi/l$.

REMARKS.

(1) Theorem 1 gives no information about the existence of real zeros in the interval (a, b). In the above mentioned paper [**1**, Theorem 2] of Karlin and Szegö it is proved that in case l odd $W(n, l; x)$ has exactly n simple zeros in (a, b).

(2) The region $S[(a, b); \pi/l]$ is *independent* of n. For example, if $l=2$, by using the Christoffel-Darboux formula, we obtain the following

COROLLARY II. *If* $\{Q_r(x)\}$ $(r=0, 1, 2, \cdots)$ *of Theorem* 1, *are also orthonormal, then all the zeros of*

$$K_n(x) = \sum_{\nu=0}^{n} [Q_\nu(x)]^2 = c_n W(n, 2; x) \qquad (n = 1, 2, \cdots) \tag{11}$$

are in the circle $|x-(a+b)/2| \leqq (b-a)/2$.

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