A CLASS OF UNIVALENT FUNCTIONS\textsuperscript{1,2}

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1. Introduction. The condition $\text{Re} \, f'(z) > 0$ is known to be sufficient for the univalence of an analytic function in any convex domain. In a recent paper [3] the author investigated the class of functions which satisfy $\text{Re} \, f'(z) > 0$ for $|z| < 1$ and are normalized by $f(0) = 0$, $f'(0) = 1$. In this paper we study the subclass denoted by $F$ and defined by the condition $|f'(z) - 1| < 1$ for $|z| < 1$. Some of our results have already been proven for the particular functions in $F$ whose coefficients satisfy $\sum_{n=2}^{\infty} n |a_n| \leq 1$. Particular reference should be made of a paper by Schild [5].

2. Distortion theorems. Suppose that $f(z) \in F$. By applying Schwarz's lemma to the function $f'(z) - 1$ we obtain $|f'(z) - 1| \leq |z|$. This gives the estimates $1 - |z| \leq |f'(z)| \leq 1 + |z|$. Bounds for $|f(z)|$ can be obtained from $|f'(z) - 1| \leq |z|$ by integration, as follows. Integrating along the line segment from $0$ to $z$ we may write

$$f(z) - z = \int_0^z \left( f'(s) - 1 \right) ds = z \int_0^1 \left( f'(tz) - 1 \right) dt.$$  

$$|f(z) - z| \leq |z| \int_0^1 |f'(tz) - 1| dt$$  

$$\leq |z| \int_0^1 t |z| dt = (1/2) |z|^2.$$  

From this estimate for $|f(z) - z|$ we immediately obtain

$$|z| - (1/2) |z|^2 \leq |f(z)| \leq |z| + (1/2) |z|^2.$$  

Each estimate is precise only for the functions $f(z) = z + a_2 z^2$, where $|a_2| = 1/2$.

The bounds for $|f(z)|$ imply the following theorem.

**Theorem 1.** Each function in $F$ assumes every complex number in the circle $|w| < 1/2$. No values outside of the circle $|w| < 3/2$ are assumed.

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3. An area theorem. If \( g(z) = \sum_{n=0}^{\infty} b_n z^n \) is regular for \( |z| < 1 \) and \( |g(z)| \leq 1 \) then \( \sum_{n=0}^{\infty} |b_n|^2 \leq 1 \) [1 p. 7]. Applying this estimate to \( f'(z) - 1 \) where \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in F \) we get \( \sum_{n=2}^{\infty} n^2 |a_n|^2 \leq 1 \). We will use this estimate in the next theorem. It also shows that the coefficients of functions in \( F \) satisfy \( |a_n| \leq 1/n \) for \( n = 2, 3, \ldots \) and equality for a given \( n \) holds only for functions of the form \( f(z) = z + a_n z^n \).

Theorem 2. The area of the image of \( |z| < 1 \) under each function in \( F \) satisfies \( A \leq (3/2)\pi \).

Proof. Suppose that \( f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in F \) and \( 0 < r < 1 \). Let \( D_r \) and \( D \) be the images of \( |z| < r \) and \( |z| < 1 \) under \( f(z) \) and let \( A_r \) and \( A \) be the areas of \( D_r \) and \( D \). The area of \( D \) exists since \( D \) is open and bounded. \( A_r \) is given by the following well-known formula.

\[
A_r = \pi \left( r^2 + \sum_{n=2}^{\infty} n |a_n|^2 r^{2n}\right).
\]

We will prove that

\[
(1) \quad A = \pi \left( 1 + \sum_{n=2}^{\infty} n |a_n|^2 \right).
\]

The convergence of this series follows from the fact that \( \sum_{n=2}^{\infty} n^2 |a_n|^2 \) converges. Let \( \{r_n\} \) be any increasing sequence of positive numbers such that \( r_n \rightarrow 1 \). Then \( \{D_{r_n}\} \) is an increasing sequence of sets whose union is \( D \). From the theorem in measure theory \( A_{r_n} \rightarrow A \). However the convergence of \( (1) \) implies that \( A_r \) is continuous for \( 0 \leq r \leq 1 \). Consequently \( A_{r_n} \rightarrow A_1 \). This proves \( (1) \).

From \( (1) \) and \( \sum_{n=2}^{\infty} n^2 |a_n|^2 \leq 1 \), we obtain

\[
A = \pi \left( 1 + \sum_{n=2}^{\infty} n |a_n|^2 \right) \\
\leq \pi \left( 1 + (1/2) \sum_{n=2}^{\infty} n^2 |a_n|^2 \right) \\
\leq (3/2)\pi.
\]

This proves \( A \leq (3/2)\pi \) and \( A = (3/2)\pi \) only for the functions \( f(z) = z + a_2 z^2 \) where \( |a_2| = 1/2 \).

4. The boundary of the image domain.

Theorem 3. Each function in \( F \) maps \( |z| < 1 \) onto a domain whose boundary is a rectifiable Jordan curve.
Proof. If \( f(z) \in F \) then \( |f'(z)| < 2 \). From \( f(z_2) - f(z_1) = \int_{z_1}^{z_2} f'(z)dz \) we obtain \( |f(z_2) - f(z_1)| \leq 2|z_2 - z_1| \). This implies that \( f(z) \) is uniformly continuous in \( |z| < 1 \) and consequently can be extended continuously onto \( |z| = 1 \). Let \( C \) be defined by \( w = f(e^{i\theta}) \), \( 0 \leq \theta \leq 2\pi \).

Let us prove that \( C \) is rectifiable. This follows easily from the estimate \( |f(z_2) - f(z_1)| \leq 2|z_2 - z_1| \) for \( |z_1| = |z_2| = 1 \). Namely, if \( 0 = \theta_0 < \theta_1 < \cdots < \theta_n = 2\pi \) then

\[
\sum_{k=1}^{n} |f(e^{i\theta_k}) - f(e^{i\theta_{k-1}})| \leq 2 \sum_{k=1}^{n} |e^{i\theta_k} - e^{i\theta_{k-1}}| < 4\pi.
\]

Next we show that each function \( f(z) \) in \( F \) is univalent in \( |z| \leq 1 \). This follows from the facts: \( \text{Re} f'(z) > 0 \) for \( |z| < 1 \) and \( f(z) \) is continuous in \( |z| \leq 1 \). It suffices to prove that \( f(z) \) is univalent on \( |z| = 1 \). Suppose that \( z_1 \neq z_2 \), \( |z_1| = |z_2| = 1 \). Let \( l \) denote the line segment from \( z_1 \) to \( z_2 \). We will consider several points, denoted by \( z_1, z_2, z_3, z_4, z_6, z_8, z_2 \). These will be distinct points on \( l \) and arranged in the order as written. Fix \( z_4 \) and then choose \( z_3 \) and \( z_6 \) such that \( \text{Re} f'(z) \geq (1/2) \text{Re} f'(z_4) \) for all points on \( l \) between \( z_3 \) and \( z_6 \). Then, integrating along \( l \) we obtain

\[
|f(z_2) - f(z_1)| = \left| \int_{z_1}^{z_2} f'(z)dz \right| \\
\geq (1/2) |z_6 - z_3| \text{ Re } f'(z_4) = a.
\]

With \( z_3, z_4, z_8 \) fixed we can choose \( z_1 \) and \( z_2 \) so close to \( z_1 \) and \( z_2 \) that

\[
|f(z_2) - f(z_3) - (f(z_1) - f(z_1))| < a.
\]

Therefore, \( f(z_2) - f(z_1) \neq 0 \). This proves that \( f(z) \) is univalent in \( |z| \leq 1 \). In particular this shows that \( C \) is simple.

Theorem 4. Suppose \( 0 < r < 1 \). The length of the image of \( |z| = r \) under functions in \( F \) is maximal for \( f_0(z) = z + (1/2)z^2 \). Moreover, this length is less than 8.

Proof. Suppose that \( f(z) \in F \) and \( L_r(f) \) is the length of the image of \( |z| = r \) under \( f(z) \). Since \( |f'(z) - 1| < 1 \) and \( f''(z) - 1 \) vanishes at \( z = 0 \), \( f'(z) \) is subordinate to \( f_0'(z) = 1 + z \) in \( |z| < 1 \). J. E. Littlewood [2, p. 484, Theorem 2] has shown that if \( h(z) \) is subordinate to \( H(z) \) in \( |z| < 1 \) then

\[
\int_{0}^{2\pi} |h(re^{i\theta})|^k d\theta \leq \int_{0}^{2\pi} |H(re^{i\theta})|^k d\theta,
\]
for any $k > 0$. Applying this result to $f'(z)$ for $k = 1$ we prove one part of the theorem, as follows.

$$L_r(f) = r \int_0^{2\pi} |f'(re^{i\theta})| \, d\theta \leq r \int_0^{2\pi} |f''(re^{i\theta})| \, d\theta = L_r(f_0).$$

$L_r(f) < 8$ will hold if we prove $L_r(f_0) < 8$. If $R \geq 1$ then $f'(z) = 1 + z$ is subordinate to $w = 1 + Rz$ in $|z| < 1$. With $R = 1/r$ we again apply Littlewood's inequality.

$$L_r(f_0) < \int_0^{2\pi} |f'(re^{i\theta})| \, d\theta \leq \int_0^{2\pi} \left| 1 + e^{i\theta} \right| \, d\theta = 2 \int_0^{2\pi} \left| \cos \left( \theta/2 \right) \right| \, d\theta = 8.$$

**Remarks.**
1. It seems likely that the length of the boundary of the image of $|z| < 1$ under each function in $F$ satisfies $L \leq 8$. This could not be improved since $L = 8$ for the functions $f(z) = z + az^n$, $|a_n| = 1/r$, $n = 2, 3, \ldots$.
2. Littlewood's inequality for $k = 2$ gives another proof of the estimate $A \leq (3/2)\pi$ in Theorem 2.

5. Radii of convexity and starlikeness.

**Theorem 5.** Each function in $F$ maps $|z| < 1/2$ onto a convex domain.

**Proof.** Suppose that $f(z) \in F$. Then $f'(z) - 1 = zg(z)$, where $g(z)$ is regular for $|z| < 1$ and $|g(z)| \leq 1$. For such functions we have the estimate

$$\left| g'(z) \right| \leq \frac{1 - \left| g(z) \right|^2}{1 - \left| z \right|^2}$$

[4, p. 168]. Also,

$$\frac{f''(z)}{f'(z)} = \frac{g(z) + zg'(z)}{1 + zg(z)}.$$  

Using the triangle inequalities and then the estimate for $|g'(z)|$ we find that

$$\left| \frac{f''(z)}{f'(z)} \right| \leq \frac{\left| g(z) \right| + \left| z \right|}{1 - \left| z \right|^2}.$$
Since $|g(z)| \leq 1$ we obtain

$$\left| \frac{f''(z)}{f'(z)} \right| \leq \frac{1}{1 - |z|}.$$

The condition $\Re\{zf''(z)/f'(z) + 1\} > 0$ for $|z| < r$ is necessary and sufficient for $f(z)$ to map $|z| < r$ onto a convex domain. This condition is satisfied if $|zf''(z)/f'(z)| < 1$. Thus, $f(z)$ is convex if $|z|/(1 - |z|) < 1$, i.e., if $|z| < 1/2$.

It is not difficult to see that $f(z) = z + a_2 z^2$, $|a_2| = 1/2$, are the only functions in $F$ which are not convex in $|z| < r$ for some $r > 1/2$.

**Theorem 6.** Each function in $F$ maps $|z| < (4/5)^{1/2}$ onto a domain which is starlike with respect to the origin.

**Proof.** The condition $\Re\{zf''(z)/f'(z)\} > 0$ for $|z| < r$ is necessary and sufficient for $f(z)$ to be starlike in $|z| < r$. In §2 we proved that $|f'(z) - 1| \leq |z|$ and $|f(z) - z| \leq (1/2) |z|^2$ if $f(z) \in F$. Taking advantage of the geometric location of $f'(z)$ and $f(z)/z$ as given by these inequalities we obtain

$$\arg f'(z) \leq \sin^{-1} |z|,$$

$$\arg \frac{f(z)}{z} \leq \sin^{-1} (|z|/2).$$

If $|z| < (4/5)^{1/2}$ then $\sin^{-1} |z| + \sin^{-1}(|z|/2) < \pi/2$. Therefore, for $|z| < (4/5)^{1/2}$ we have

$$\left| \arg \frac{zf''(z)}{f'(z)} \right| \leq \left| \arg f'(z) \right| + \left| \arg \frac{f(z)}{z} \right| < \pi/2,$$

i.e., $\Re\{zf''(z)/f'(z)\} > 0$ for $|z| < (4/5)^{1/2}$.

The estimates used in this proof are precise only for the functions $f(z) = z + a_2 z^2$, where $|a_2| = 1/2$. Since these functions map the whole circle $|z| < 1$ onto a starlike domain, $(4/5)^{1/2}$ is not the radius of starlikeness for the class $F$.

**6. Functions with initial zero coefficients.** Some of the results obtained for functions in $F$ can be improved if $f(z)$ has the form $f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \cdots$. In this situation we can write $f'(z) - 1 = z^{n-1} g(z)$, where $g(z)$ is regular for $|z| < 1$ and $|g(z)| \leq 1$. With this as the starting point we can argue as in §§2 and 3 to prove the following theorem. The extremal functions for this theorem are $f(z) = z + a_n z^n$, where $|a_n| = 1/n$. 
**Theorem 7.** Suppose that \( f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \cdots \in F \). Then \( f(z) \) assumes all values in the circle \( |w| < 1 - (1/n) \). No values outside of the circle \( |w| < 1 + (1/n) \) are assumed. The area of the image domain satisfies \( A \leq \pi (1 + 1/n) \).

**Theorem 8.** If \( f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \cdots \in F \) then \( f(z) \) maps \( |z| < (1/n)^{1/(n-1)} \) onto a convex domain.

**Proof.** From \( f'(z) - 1 = z^{-1} g(z) \) it follows that
\[
\frac{f''(z)}{f'(z)} = \frac{n^2}{1 + z^{n-1} g(z)} g(z) + z g'(z).
\]
Using the triangle inequalities and then the estimate \( |g'(z)| \leq (1 - |g(z)|^2)/(1 - |z|^2) \), we obtain
\[
\left| \frac{f''(z)}{f'(z)} \right| \leq \frac{|z|^{n-2} + (n-1)(1 - |z|) |g(z)| - |z| |g(z)|^2}{1 - |z|^{n-1} |g(z)|}.
\]
To estimate the right side of this inequality let us consider the function
\[
y = a + (n-1)(1 - a^2) x - a x^2,
\]
\[
a = |z|, \quad x = |g(z)|, \quad 0 < a < 1, \quad 0 \leq x \leq 1.
\]
\[
p = (1 - a^{n-1} x) \frac{dy}{dx} = (n-1)(1 - a^2) + a^2 - 2 ax + a^2 x^2
\]
\[
\frac{dp}{dx} = 2a(a^{n-1}x - 1) < 0.
\]
Thus \( p \) is decreasing.
\[
p(x) \geq p(1) = (n-1)(1 - a^2) - 2a(1 - a^{n-1})
\]
\[
= (1 - a)[(n-1)(1 + a) - 2a(1 + a + a^2 + \cdots + a^{n-2})]
\]
\[
> (1 - a)[(n-1)(1 + a) - 2a(n - 1)]
\]
\[
= (n-1)(1 - a)^2 > 0.
\]
Thus \( y \) is an increasing function. Therefore
\[
y(x) \leq y(1) = \frac{(n-1)(1 - a^2)}{1 - a^{n-1}},
\]
\[
\left| \frac{f''(z)}{f'(z)} \right| \leq \frac{(n-1) |z|^{n-2}}{1 - |z|^{n-1}}.
\]
If \( |zf''(z)/f'(z)| < 1 \) then \( f(z) \) is convex. Thus, \( f(z) \) is convex if 
\[
(n-1) \frac{|z|^{n-1}}{1-|z|^{n-1}} < 1, \text{ i.e., if } |z| < \left(\frac{1}{n}\right)^{1/(n-1)}.
\]

The functions \( f(z) = z + a_n z^n, \) \( |a_n| = 1/n, \) are extremal, i.e., they
are the only functions in \( F \) of the form \( f(z) = z + a_n z^n + a_{n+1} z^{n+1} + \cdots \)
which are not convex in some circle \( |z| < r, r > \left(\frac{1}{n}\right)^{1/(n-1)} \).

**References**


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