CONTRACTION OF WALSH FOURIER SERIES

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The purpose of this note is to prove an analogue for Walsh Fourier series (abbreviated WFS) of a theorem of R. P. Boas [1]. For basic properties of Walsh functions, the reader is referred to N. J. Fine [2].

We begin with some notations and definitions:

The Walsh functions are denoted by $\psi_n(x)$ $(n=0, 1, 2, \cdots)$ and considered in the "real version."

A complex-valued function $T(z)$ of a complex variable z is called a *contraction* if it satisfies the inequality

$$|T(z_1) - T(z_2)| \leqq |z_1 - z_2|.$$

An integrable function $f(x)$[1] is *contractible* if it has absolutely convergent WFS and the same is true for $T(f(x))$ where T is any contraction.

Our theorem now reads as follows:

THEOREM. *Let* $\{\omega_n\}$ *be a sequence of non-negative numbers with* $\sum_{n=1}^{\infty} \omega_n < \infty$ *and*

$$\sum_{n=1}^{\infty} n^{-3/2}\left(\sum_{\nu=1}^{n} \nu^2 \omega_\nu^2\right)^{1/2} + \sum_{n=1}^{\infty} n^{-1/2}\left(\sum_{\nu=n}^{\infty} \omega_\nu^2\right)^{1/2} < \infty. \tag{1}$$

Then an integrable function $f(x)$ *with*

$$f(x) \sim \sum_{n=0}^{\infty} c_n \psi_n(x), \qquad |c_n| \leqq \omega_n$$

is contractible.

The proof of this theorem is essentially the same as Boas', the only difference being Lemma 3 below, in which an elementary property of Walsh function is used.

LEMMA 1. *For a non-negative sequence* $\{\omega_n\}$, (1) *is equivalent to*

$$\sum_{n=1}^{\infty} 2^{-n/2}\left(\sum_{\nu=1}^{2^n-1} 2^{2k} \omega_\nu^2\right)^{1/2} + \sum_{n=1}^{\infty} 2^{n/2}\left(\sum_{\nu=2^n}^{\infty} \omega_\nu^2\right)^{1/2} < \infty$$

where $k=k(\nu)$ *is the integer satisfying* $2^k \leqq \nu < 2^{k+1}$.

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[1] We consider measurable functions with period 1 only; thus the integrability is meant over an interval of length 1, say [0, 1).

PROOF. The equivalence between

$$\sum_{n=1}^{\infty} n^{-1/2}\left(\sum_{\nu=n}^{\infty} \omega_\nu^2\right)^{1/2} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} 2^{n/2}\left(\sum_{\nu=2^n}^{\infty} \omega_\nu^2\right)^{1/2} < \infty$$

is nothing but Cauchy's condensation theorem. On the other hand,

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{-3/2}\left(\sum_{\nu=1}^{n} \nu^2 \omega_\nu^2\right)^{1/2} \\
&= \sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} n^{-3/2}\left(\sum_{\nu=1}^{n} \nu^2 \omega_\nu^2\right)^{1/2} \leqq \sum_{j=0}^{\infty} 2^{-3j/2} \sum_{n=2^j}^{2^{j+1}-1} \left(\sum_{\nu=1}^{n} \nu^2 \omega_\nu^2\right)^{1/2} \\
&\leqq A \sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} 2^{-3j/2}\left(\sum_{\nu=1}^{2^{j+1}-1} 2^{2k} \omega_\nu^2\right)^{1/2} = A \sum_{j=0}^{\infty} 2^{-j/2}\left(\sum_{\nu=1}^{2^{j+1}-1} 2^{2k} \omega_\nu^2\right)^{1/2} \\
&\leqq A \sum_{j=1}^{\infty} 2^{-j/2}\left(\sum_{\nu=1}^{2^{j-1}} 2^{2k} \omega_\nu^2\right)^{1/2},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} n^{-3/2}\left(\sum_{\nu=1}^{n} \nu^2 \omega_\nu^2\right)^{1/2} &\geqq A \sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} 2^{-3j/2}\left(\sum_{\nu=1}^{n} 2^{2k} \omega_\nu^2\right)^{1/2} \\
&\geqq A \sum_{j=0}^{\infty} 2^{-3j/2} \sum_{n=2^j}^{2^{j+1}-1} \left(\sum_{\nu=1}^{2^{j-1}} 2^{2k} \omega_\nu^2\right)^{1/2} \\
&= A \sum_{j=0}^{\infty} 2^{-j/2}\left(\sum_{\nu=1}^{2^{j-1}} 2^{2k} \omega_\nu^2\right)^{1/2}, \quad \text{q.e.d.}
\end{aligned}$$

LEMMA 2. *For any non-negative sequence* $\{a_\nu\}$,

$$\sum_{\nu=1}^{\infty} a_\nu \leqq \sum_{n=1}^{\infty} 2^{-n/2}\left(\sum_{\nu=1}^{2^n-1} 2^{2k} a_\nu^2\right)^{1/2}.$$

PROOF.

$$\begin{aligned}
\sum_{\nu=1}^{\infty} a_\nu &= \sum_{j=0}^{\infty} \sum_{\nu=2^j}^{2^{j+1}-1} a_\nu = \sum_{j=0}^{\infty} 2^j \sum_{\nu=2^j}^{2^{j+1}-1} a_\nu \sum_{n=j+1}^{\infty} 2^{-n} \\
&= \sum_{n=1}^{\infty} 2^{-n} \sum_{j=0}^{n-1} 2^j \sum_{\nu=2^j}^{2^{j+1}-1} a_\nu = \sum_{n=1}^{\infty} 2^{-n} \sum_{j=0}^{n-1} \sum_{\nu=2^j}^{2^{j+1}-1} 2^k a_\nu \\
&= \sum_{n=1}^{\infty} 2^{-n} \sum_{\nu=1}^{2n-1} 2^k a_\nu \leqq \sum_{n=1}^{\infty} 2^{-n} \cdot 2^{n/2}\left(\sum_{\nu=1}^{2^n-1} 2^{2k} a_\nu^2\right)^{1/2} \\
&= \sum_{n=1}^{\infty} 2^{-n/2}\left(\sum_{\nu=1}^{2n-1} 2^{2k} a_\nu^2\right)^{1/2}, \quad \text{q.e.d}
\end{aligned}$$

LEMMA 3. *Let* $f(x)=\sum_{\nu=0}^{\infty} c_\nu\psi_\nu(x)$ *with* $\sum_{\nu=0}^{\infty} |c_\nu|^2<\infty$ *and let* $g(x) \sim \sum_{\nu=0}^{\infty} a_\nu\psi_\nu(x)$ *be a contraction of* $f(x)$. *Then*

$$\sum_{\nu=1}^{2^n-1} 2^{2k}|a_\nu|^2 \leqq A\sum_{\nu=1}^{2^n-1} 2^{2k}|c_\nu|^2 + A\,2^n\sum_{\nu=2^n}^{\infty}|c_\nu|^2.$$

PROOF. For any h, $0<h<1$, choose the integer m so that

$$2^{-m-1} \leqq h < 2^{-m}.$$

We have, by the definition of Walsh functions,

$$\psi_\nu(h) = -1 \qquad (2^m \leqq \nu < 2^{m+1}). \tag{2}$$

From Parseval's relation combined with (2), it follows that

$$\begin{aligned} 4\sum_{\nu=2^m}^{2^{m+1}-1}|a_\nu|^2 &\leqq \sum_{\nu=2^m}^{\infty}|a_\nu|^2|1-\psi_\nu(h)|^2 = \int_0^1 |g(x\dotplus h)-g(x)|^2\,dx \\ &\leqq \int_0^1 |f(x\dotplus h)-f(x)|^2dx = \sum_{\nu=2^m}^{\infty}|c_\nu|^2|1-\psi_\nu(h)|^2 \\ &\leqq 4\sum_{\nu=2^m}^{\infty}|c_\nu|^2. \end{aligned}$$

Thus we have

$$\sum_{\nu=2^m}^{2^{m+1}-1}|a_\nu|^2 \leqq \sum_{\nu=2^m}^{\infty}|c_\nu|^2 = \sum_{j=m}^{\infty}\sum_{\nu=2^j}^{2^{j+1}-1}|c_\nu|^2. \tag{3}$$

Since (3) is true for each m, multiplication by 2^{2m} and summation with respect to m give

$$\begin{aligned} \sum_{m=0}^{n-1}2^{2m}\sum_{\nu=2^m}^{2^{m+1}-1}|a_\nu|^2 &= \sum_{m=0}^{n-1}\sum_{\nu=2^m}^{2^{m+1}-1}2^{2k}|a_\nu|^2 \leqq \sum_{m=0}^{n-1}2^{2m}\sum_{j=m}^{\infty}\sum_{\nu=2^j}^{2^{j+1}-1}|c_\nu|^2 \\ &= \sum_{m=0}^{n-1}2^{2m}\sum_{j=m}^{n-1}\sum_{\nu=2^j}^{2^{j+1}-1}|c_\nu|^2 + \sum_{m=0}^{n-1}2^{2m}\sum_{j=n}^{\infty}\sum_{\nu=2^j}^{2^{j+1}-1}|c_\nu|^2 \\ &= \sum_{j=0}^{n-1}\sum_{\nu=2^j}^{2^{j+1}-1}|c_\nu|^2\sum_{m=0}^{j}2^{2m} + \sum_{j=n}^{\infty}\sum_{\nu=2^j}^{2^{j+1}-1}|c_\nu|^2\sum_{m=0}^{n-1}2^{2m} \\ &\leqq A\sum_{j=0}^{n-1}2^{2j}\sum_{\nu=2^j}^{2^{j+1}-1}|c_\nu|^2 + A\cdot 2^{2n}\sum_{j=n}^{\infty}\sum_{\nu=2^j}^{2^{j+1}-1}|c_\nu|^2 \\ &\leqq A\sum_{\nu=1}^{2^n-1}2^{2k}|c_\nu|^2 + A\cdot 2^{2n}\sum_{\nu=2^n}^{\infty}|c_\nu|^2, \quad \text{q.e.d.} \end{aligned}$$

PROOF OF THEOREM. By Lemmas 2 and 3, we have

$$\begin{aligned}\sum_{\nu=1}^{\infty} |a_\nu| &\leqq A \sum_{n=1}^{\infty} 2^{-n/2}\left(\sum_{\nu=1}^{2^n-1} 2^{2k}|a_\nu|^2\right)^{1/2} \\ &\leqq A \sum_{n=1}^{\infty} 2^{-n/2}\left(\sum_{\nu=1}^{2^n-1} 2^{2k}|c_\nu|^2 + 2^{2n}\sum_{\nu=2^n}^{\infty}|c_\nu|^2\right)^{1/2} \\ &\leqq A \sum_{n=1}^{\infty} 2^{-n/2}\left(\sum_{\nu=1}^{2^n-1} 2^{2k}\omega_\nu^2\right)^{1/2} + A\sum_{n=1}^{\infty} 2^{n/2}\left(\sum_{\nu=2^n}^{\infty}\omega_\nu^2\right)^{1/2},\end{aligned}$$

which is convergent by Lemma 1 and the assumption of Theorem.

REFERENCES

1. R. P. Boas, *Beurling's test for absolute convergence of Fourier series*, Bull. Amer. Math. Soc. **66** (1960), 24–27.

2. N. J. Fine, *On Walsh functions*, Trans. Amer. Math. Soc. **65** (1949), 372–414.

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