CONTRACTION OF WALSH FOURIER SERIES

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The purpose of this note is to prove an analogue for Walsh Fourier series (abbreviated WFS) of a theorem of R. P. Boas [1]. For basic properties of Walsh functions, the reader is referred to N. J. Fine [2].

We begin with some notations and definitions:

The Walsh functions are denoted by $\psi_n(x)$ ($n=0, 1, 2, \cdots$) and considered in the "real version."

A complex-valued function $T(z)$ of a complex variable $z$ is called a contraction if it satisfies the inequality

$$|T(z_1) - T(z_2)| \leq |z_1 - z_2|.$$ 

An integrable function $f(x)^1$ is contractible if it has absolutely convergent WFS and the same is true for $T(f(x))$ where $T$ is any contraction.

Our theorem now reads as follows:

**Theorem.** Let $\{\omega_n\}$ be a sequence of non-negative numbers with $\sum_{n=1}^{\infty} \omega_n < \infty$ and

$$\sum_{n=1}^{\infty} n^{-3/2} \left( \sum_{j=1}^{n} 2^j \omega_j \right)^{1/2} + \sum_{n=1}^{\infty} n^{-1/2} \left( \sum_{j=n}^{\infty} 2^j \omega_j \right)^{1/2} < \infty.$$ 

Then an integrable function $f(x)$ with

$$f(x) \sim \sum_{n=0}^{\infty} c_n \psi_n(x), \quad |c_n| \leq \omega_n$$

is contractible.

The proof of this theorem is essentially the same as Boas', the only difference being Lemma 3 below, in which an elementary property of Walsh function is used.

**Lemma 1.** For a non-negative sequence $\{\omega_n\}$, (1) is equivalent to

$$\sum_{n=1}^{\infty} 2^{-n/2} \left( \sum_{i=1}^{2^{n-1}} 2^{2i} \omega_i \right)^{1/2} + \sum_{n=1}^{\infty} 2^{-n/2} \left( \sum_{i=n}^{\infty} 2^i \omega_i \right)^{1/2} < \infty$$

where $k = k(\nu)$ is the integer satisfying $2^k \leq \nu < 2^{k+1}$.

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^1 We consider measurable functions with period 1 only; thus the integrability is meant over an interval of length 1, say $[0, 1)$. 189
PROOF. The equivalence between
\[ \sum_{n=1}^{\infty} n^{-1/2} \left( \sum_{r=n}^{\infty} \frac{1}{\omega_r} \right)^{1/2} < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} 2^{n/2} \left( \sum_{r=2^n}^{\infty} \frac{1}{\omega_r} \right)^{1/2} < \infty \]

is nothing but Cauchy's condensation theorem. On the other hand,
\[
\sum_{n=1}^{\infty} n^{-3/2} \left( \sum_{r=1}^{n} \frac{2}{\omega_r} \right)^{1/2} \\
= \sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} n^{-3/2} \left( \sum_{r=1}^{n} \frac{2}{\omega_r} \right)^{1/2} \leq \sum_{j=0}^{\infty} 2^{-3j/2} \sum_{n=2^j}^{2^{j+1}-1} \left( \sum_{r=1}^{n} \frac{2}{\omega_r} \right)^{1/2} \\
\leq A \sum_{j=0}^{\infty} 2^{-j/2} \left( \sum_{r=1}^{2^{j+1}-1} 2k \frac{2}{\omega_r} \right)^{1/2} = A \sum_{j=0}^{\infty} 2^{-j/2} \left( \sum_{r=1}^{2^{j+1}-1} 2k \frac{2}{\omega_r} \right)^{1/2} \\
\leq A \sum_{j=0}^{\infty} 2^{-j/2} \left( \sum_{r=1}^{2^{j+1}-1} 2k \frac{2}{\omega_r} \right)^{1/2},
\]

and
\[
\sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} n^{-3/2} \left( \sum_{r=1}^{n} \frac{2}{\omega_r} \right)^{1/2} \geq A \sum_{j=0}^{\infty} \sum_{n=2^j}^{2^{j+1}-1} 2^{-3j/2} \left( \sum_{r=1}^{n} 2k \frac{2}{\omega_r} \right)^{1/2} \\
\geq A \sum_{j=0}^{\infty} 2^{-3j/2} \sum_{n=2^j}^{2^{j+1}-1} \left( \sum_{r=1}^{n} 2k \frac{2}{\omega_r} \right)^{1/2} \\
= A \sum_{j=0}^{\infty} 2^{-j/2} \left( \sum_{r=1}^{2^{j+1}-1} 2k \frac{2}{\omega_r} \right)^{1/2}, \quad \text{q.e.d.}
\]

**Lemma 2.** For any non-negative sequence \( \{a_r\} \),
\[
\sum_{r=1}^{\infty} a_r \leq \sum_{n=1}^{\infty} 2^{-n/2} \left( \sum_{r=1}^{2^n-1} 2k a_r \right)^{1/2}.
\]

**Proof.**
\[
\sum_{r=1}^{\infty} a_r = \sum_{j=0}^{\infty} \sum_{r=2^j}^{2^{j+1}-1} a_r = \sum_{j=0}^{\infty} 2^j \sum_{r=2^j}^{2^{j+1}-1} a_r = \sum_{j=0}^{\infty} 2^j \sum_{r=2^j}^{2^{j+1}-1} 2k a_r \\
= \sum_{n=1}^{\infty} 2^{-n} \sum_{r=1}^{2^n-1} 2k a_r \leq \sum_{n=1}^{\infty} 2^{-n} \cdot 2^n \left( \sum_{r=1}^{2^n-1} 2k a_r \right)^{1/2} \\
= \sum_{n=1}^{\infty} 2^{-n/2} \left( \sum_{r=1}^{2^n-1} 2k a_r \right)^{1/2}, \quad \text{q.e.d.}
\]
Lemma 3. Let \( f(x) = \sum_{r=0}^{\infty} c_r \psi_r(x) \) with \( \sum_{r=0}^{\infty} |c_r|^2 < \infty \) and let \( g(x) \sim \sum_{r=0}^{\infty} a_r \psi_r(x) \) be a contraction of \( f(x) \). Then

\[
\sum_{r=1}^{2^n-1} 2^{2k} |a_r|^2 \leq A \sum_{r=1}^{2^n-1} 2^{2k} |c_r|^2 + A \cdot 2^n \sum_{r=2^n}^{2^{n+1}-1} |c_r|^2.
\]

Proof. For any \( h, 0 < h < 1 \), choose the integer \( m \) so that

\[
2^{-m-1} \leq h < 2^{-m}.
\]

We have, by the definition of Walsh functions,

\[
(2) \quad \psi_r(h) = -1 \quad (2^m \leq r < 2^{m+1}).
\]

From Parseval's relation combined with (2), it follows that

\[
4 \sum_{r=2^m}^{2^{m+1}-1} |a_r|^2 \leq \sum_{r=2^m}^{2^{m+1}-1} |a_r|^2 \sum_{r=2^m}^{2^{m+1}-1} 1 - \psi_r(h)^2 = \int_{0}^{1} \left| g(x + h) - g(x) \right|^2 dx
\]

\[
\leq \int_{0}^{1} \left| f(x + h) - f(x) \right|^2 dx = \sum_{r=2^m}^{2^{m+1}-1} |c_r|^2 \sum_{r=2^m}^{2^{m+1}-1} 1 - \psi_r(h)^2
\]

\[
\leq 4 \sum_{r=2^m}^{2^{m+1}-1} |c_r|^2.
\]

Thus we have

\[
(3) \quad \sum_{r=2^m}^{2^{m+1}-1} |a_r|^2 \leq \sum_{r=2^m}^{2^{m+1}-1} |c_r|^2 = \sum_{j=m}^{\infty} \sum_{r=2^j}^{2^{j+1}-1} |c_r|^2.
\]

Since (3) is true for each \( m \), multiplication by \( 2^{2m} \) and summation with respect to \( m \) give

\[
\sum_{m=0}^{n-1} 2^{2m} \sum_{r=2^m}^{2^{m+1}-1} |a_r|^2 = \sum_{m=0}^{n-1} 2^{2m} \sum_{r=2^m}^{2^{m+1}-1} 2^{2k} |a_r|^2 \leq \sum_{m=0}^{n-1} 2^{2m} \sum_{j=m}^{\infty} \sum_{r=2^j}^{2^{j+1}-1} |c_r|^2
\]

\[
= \sum_{m=0}^{n-1} 2^{2m} \sum_{j=m}^{\infty} \sum_{r=2^j}^{2^{j+1}-1} |c_r|^2 + \sum_{m=0}^{n-1} 2^{2m} \sum_{j=n}^{\infty} \sum_{r=2^{j+n}}^{2^{j+1}-1} |c_r|^2
\]

\[
= \sum_{j=n}^{\infty} \sum_{r=2^j}^{2^{j+1}-1} |c_r|^2 + \sum_{j=n}^{\infty} \sum_{m=0}^{2^j} \sum_{r=2^j}^{2^{j+1}-1} |c_r|^2
\]

\[
\leq A \sum_{j=0}^{n-1} 2^{2j} \sum_{r=2^j}^{2^{j+1}-1} |c_r|^2 + A \cdot 2^{2n} \sum_{j=n}^{\infty} \sum_{r=2^j}^{2^{j+1}-1} |c_r|^2
\]

\[
\leq A \sum_{r=1}^{2^{n-1}} 2^{2k} |c_r|^2 + A \cdot 2^{2n} \sum_{r=2^n}^{2^{n+1}-1} |c_r|^2.
\]

q.e.d.
Proof of Theorem. By Lemmas 2 and 3, we have
\[
\sum_{r=1}^{\infty} |a_r| \leq A \sum_{n=1}^{\infty} 2^{-n/2} \left( \sum_{r=1}^{2^{n-1}} |a_r|^2 \right)^{1/2}
\]
\[
\leq A \sum_{n=1}^{\infty} 2^{-n/2} \left( \sum_{r=1}^{2^{n-1}} |c_r|^2 + 2^{2n} \sum_{r=2^n}^{\infty} |c_r|^2 \right)^{1/2}
\]
\[
\leq A \sum_{n=1}^{\infty} 2^{-n/2} \left( \sum_{r=1}^{2^{n-1}} 2^{2k} \omega_r \right)^{1/2} + A \sum_{n=1}^{\infty} 2^{n/2} \left( \sum_{r=2^n}^{\infty} \omega_r \right)^{1/2},
\]
which is convergent by Lemma 1 and the assumption of Theorem.

References


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