A NOTE ON SAALFRANK'S GENERALIZATION OF ABSOLUTE RETRACT

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In a recent paper [4], Saalfrank has defined the concept of absolute homotopy retract for compact Hausdorff spaces. By using a technique of Hanner [2] which avoids the necessity of a universal imbedding space, we generalize Saalfrank's results to include Lindelöf and fully normal spaces. Finally, an indication is given for the further generalization to include the metric cases.

All spaces are Hausdorff and mapping will always mean continuous function. Let Q denote either the fully normal, Lindelöf or compact class.

Definition. X is an absolute homotopy retract relative to the class $Q\{\mathrm{AHR}(Q)\}$ if $X\in Q$ and every homeomorph of X which is a subset of a Q space is a homotopy retract [4] of that space.

Definition. X is a homotopy extension space relative to the class $Q\{\mathrm{HES}(Q)\}$ if each mapping f of a closed subset B of a Q space Y into X has a homotopy extension [4] to Y.

Since all mappings into a contractible space are inessential, we have the following

Lemma. *If X is contractible, then each mapping of a subset B of Y into X has a homotopy extension to Y.*

Theorem. *The following are equivalent if $X\in Q$.* (a) *X is an* $\mathrm{AHR}(Q)$. (b) *X is an* $\mathrm{HES}(Q)$. (c) *X is contractible.*

Proof. (a)$\Rightarrow$(b). Suppose that (Y, B) is a Q pair and f is a mapping of B into X. Let Z be the identification space obtained from $X\cup Y$ by identifying $y\in B$ with $f(y)\in X$. Take $j: X\rightarrow Z$ to be given by $j(x)=x$ and $k: Y\rightarrow Z$ to be given by $k(y)=f(y)$ if $y\in B$ and $k(y)=y$ otherwise. A set $G\subset Z$ is defined to be open if and only if $j^{-1}(G)$ and $k^{-1}(G)$ are open in X and Y, respectively. $k|(Y-B)$ is a homeomorphism onto $Z-X$; hence X is closed in Z. Hanner [2] has shown that Z is a Q space for the Q classes under consideration whenever X and Y are Q spaces. Then (Z, X) is a Q pair; therefore, there exists $r: Z\rightarrow X$ continuously with $r|X\simeq i$. Since $rk: Y\rightarrow X$ and $rk|B=rf\simeq f$, $F=rk$ is the desired homotopy extension of f.

(b)$\Rightarrow$(c). If $Z=X\times I$ and $B=(X\times\{0\})\cup(X\times\{1\})$, then (Z, B) is a Q pair. Define $f: B\rightarrow X$ by $f(x, 0)=x, f(x, 1)=x_0\in X$. Then there

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exists a mapping $F: X\times I \rightarrow X$ with $F|(X\times\{0\})\simeq i$ and $F|(X\times\{1\})$ homotopic to the constant map of X onto x_0. Therefore X is contractible.

(c)$\Rightarrow$(a). According to the lemma, we can assume that X is an HES(Q). Let X_1, a homeomorph of X under h, be a closed subset of a Q space Z. Then $f=h^{-1}$ is a mapping on X_1 to X. Since X is an HES(Q), there exists a mapping $F: Z\rightarrow X$ such that $F|X\simeq f$; hence $hF|X\simeq hf = i$ and X_1 is a homotopy retract of Z.

Since the product of contractible spaces is contractible, we have

COROLLARY 1. *If each* $\{X_\alpha : \alpha \in A\}$ *is an* AHR(Q), *then* $P=\prod\{X_\alpha : \alpha\in A\}$ *is an* HES(Q).

In particular, if Q is the compact class, then P is an AHR(Q) by virtue of Tychonoff's theorem. Since all spaces in the Q classes under consideration are fully normal, we can restate a theorem of Hanner [**2**, Theorem 12.6] in terms of AHR(Q)'s.

COROLLARY 2. *X is an* AR(Q) *if and only if X is an* ANR(Q) *and an* AHR(Q).

Hausdorff [**3**; **1**] has shown that there is a metrizable topology for the imbedding space Z such that X is imbedded as a closed subset of Z and f is extendable relative to Z. If we use this topology for Z, the results of the theorem and corollaries hold for the metric classes (i.e., metric, separable metric and compact metric) as well.

Let X be the space consisting of the tangent cones and the line segment from a to b in Figure 1.

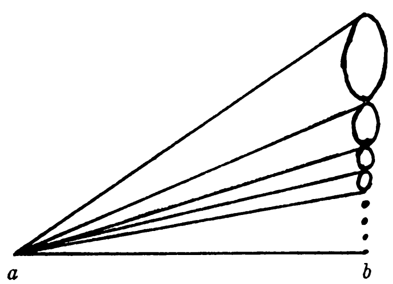

FIGURE 1

X is contractible and belongs to Q for the classes considered above; hence X is an AHR(Q) for these Q classes. However, X is not locally contractible; therefore it is not an AR(Q), or even an ANR(Q), for these classes [**2**]. This example also serves to answer the question at the end of [**4**] since X is locally connected.

References

1. Richard Arens, *Extension of functions on fully normal spaces*, Pacific J. Math. 2 (1952), 11–22.

2. O. Hanner, *Retraction and extension of mappings of metric and non-metric spaces*, Ark. Mat. 2 (1952), 315–360.

3. F. Hausdorff, *Erweiterung einer stetigen Abbildung*, Fund. Math. 30 (1938), 40–47.

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A COVERING THEOREM FOR CONVEX MAPPINGS[1]

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The following theorems are classical. Proofs can be found in [1, pp. 214, 223].

Theorem 1. *If $f(z)$ is regular and univalent in $|z|<1$, $f(0)=0$ and $f'(0)=1$ then the image domain covers the circle $|w|<1/4$.*

Theorem 2. *If $f(z)$ is regular and univalent in $|z|<1$, $f(0)=0$, $f'(0)=1$ and the image D is convex, then D covers the circle $|w|<1/2$.*

The purpose of this note is to show that Theorem 2 can be proven as a simple consequence of Theorem 1. Suppose then that $f(z)$ satisfies the hypotheses of Theorem 2 and $f(z)\neq c$. Let $g(z)=(f(z)-c)^2$. Suppose that z_1 and z_2 are distinct points in the unit circle and $g(z_1)=g(z_2)$. Then either $f(z_1)=f(z_2)$ or $2^{-1}(f(z_1)+f(z_2))=c$. The first equation cannot hold since $f(z)$ is univalent in $|z|<1$. Neither can the second equation hold, for D is convex and therefore the average of every two points in D is also in D. This proves that $g(z)$ is univalent in $|z|<1$. The function $h(z)=(c^2-g(z))/2c$ satisfies the hypotheses of Theorem 1, and $h(z)\neq c/2$ since $f(z)\neq c$. Therefore, $|c/2|\geqq 1/4$, $|c|\geqq 1/2$.

Reference

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