ON A THEOREM OF SABIDUSSI

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By a graph X we mean a finite set $V(X)$ called the vertices of X, together with a set $E(X)$, called the edges of X, consisting of unordered pairs of distinct elements of $V(X)$. We shall indicate unordered pairs by brackets. We denote $G(X)$ as the group of automorphisms of X. Each element ρ of $G(X)$ is considered as a permutation of $V(X)$ onto itself, such that $[\rho x, \rho y] \in E(X)$ if and only if $[x, y] \in E(X)$. $G(X)$ is said to be strongly fixed-point-free if $G(X) \neq \{1\}$, and $\phi x \neq x$ for every $x \in X$ and every $\phi \in G(X) - \{1\}$, where 1 is the identity of $G(X)$. $G(X)$ is said to be regular if it is strongly fixed-point-free and transitive on $V(X)$.

Let G be a finite group and H a subset of G which does not contain the identity of G. By the color-group $X_{G,H}$ of G with respect to H, we mean the graph given by $V(X_{G,H}) = G$ and $E(X_{G,H}) = \{ [g, gh]; g \in G, h \in H\}$. It is known that $X_{G,H}$ is connected if and only if H contains a set of generators of G.

Recently, Sabidussi showed [**2**, p. 802] the following theorem:

Let $G(X)$ act regularly on a graph X, then X is isomorphic to either the graph Y_0 with $V(Y_0) = \{x_1, x_2\}$, $E(Y_0) = \square$, or X is connected and isomorphic to a color-group of $G(X)$ with respect to some set H of generators of $G(X)$.

The purpose of this note is to prove the following theorem by using Sabidussi's result.

THEOREM. *There exists no graph with n vertices whose group of automorphisms is transitive and abelian where $n > 2$.*

PROOF. We know that every transitive and abelian permutation group is regular (see [**3**, p. 12]). By Sabidussi's theorem we only need to consider the connected graphs.

Suppose there existed a connected graph X with n vertices where $n > 2$ such that $G(X)$ is regular and abelian, then we would have the order and degree of $G(X)$ equal to n. Let $G(X)$ be denoted by $\{g_0 = 1, g_1, \cdots, g_{n-1}\}$. By Sabidussi's theorem, there exists a set $H = \{g_{k_1}, g_{k_2}, \cdots, g_{k_t}\}$ consisting of generators of $G(X)$, in $G(X)$ where $1 \leqq t \leqq n-1$, and $1 \leqq k_i \leqq n-1$ for $i = 1, 2, \cdots, t$ such that $X \cong X_{G(X),H}$. That is $V(X_{G(X),H}) = \{g_0, g_1, \cdots, g_{n-1}\}$ and $E(X_{G(X),H}) = \{ [g_i, g_i g_{k_j}];$

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$i=0, 1, \cdots, n-1$, and $j=1, 2, \cdots, t\}$. That means the edges incident with each g_i are $[g_i, g_ig_{k_j}]=[g_i, g_{k_j}g_i]$ and $[g_ig_{k_j}^{-1}, g_i]=[g_{k_j}^{-1}g_i, g_i]$ for $i=0, 1, \cdots, n-1$, and $j=1, 2, \cdots, t$ where commutativity is used.

Let ρ be a permutation of $V(X_{G(X),H})$ defined by $\rho(g_i)=g_i^{-1}$, $i=0, 1, \cdots, n-1$. Since $[\rho(g_i), \rho(g_ig_{k_j})]=[g_i^{-1}, g_{k_j}^{-1}g_i^{-1}]$ and $[\rho(g_ig_{k_j}^{-1}), \rho(g_i)]=[g_{k_j}g_i^{-1}, g_i^{-1}]$ belong to $E(X_{G(X),H})$ if and only if $[g_i, g_ig_{k_j}]$ and $[g_ig_{k_j}^{-1}, g_i]$ belong to $E(X_{G(X),G})$, ρ is an automorphism of $X_{G(X),H}$. Since $G(X)$ is strongly fixed-point-free, $\rho=1$. Hence, every nonidentity element in $G(X)$ is of order two, and $G(X)\cong C_2\times C_2 \times \cdots \times C_2$, where C_2 is the cyclic group of order 2. Since $n>2$, the number of generators in H must be >1. Consequently, there is a nonidentity permutation σ of $V(X_{G(X),H})$ which leaves g_0 fixed and $\sigma H=H$. Since each element of $G(X)$ can be written as $g_{k_1}^{\lambda_1}g_{k_2}^{\lambda_2}\cdots g_{k_m}^{\lambda_m}$ where g_{k_i} are generators of $G(X)$, $m>1$, and $\lambda_i=0$ or 1 $(i=1, 2, \cdots, m)$, and since every such product is an element of $G(X)$, it is easy to see that σ is an automorphism of $X_{G(X),H}$. Since $\sigma(g_0)=g_0$ and $X_{G(X),H}\cong X$, $G(X)$ is not strongly fixed-point-free. That is a contradiction and the proof is completed.

COROLLARY 1 (KAGNO [1]). *There exists no graph with n vertices whose group of automorphisms is the cyclic group generated by the cycle $(1\ 2 \cdots n)$ for $n>2$.*

PROOF. Since the cyclic group generated by the cycle $(1\ 2\cdots n)$ is a transitive and abelian group, apply the theorem to complete the proof.

COROLLARY 2. *There exists no graph with n vertices whose group of automorphisms is regular where n is an odd prime.*

PROOF. Since n is an odd prime and since the order of $G(X)$ is n, $G(X)$ is a cyclic group. Apply Corollary 1 to complete the proof.

I am grateful to Professor Sabidussi for pointing out Kagno's paper, and for putting the theorem in present form.

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