ON COMPLETELY DECOMPOSABLE GROUPS

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In this note I shall give some theorems concerning completely decomposable groups. I use the same definition and follow the same notations as those in the book [1].

DEFINITION. A subgroup H of a torsion free group G is said to be regular if the elements of H have the same types in H and G.

LEMMA 1. *If H is a regular subgroup of a torsion free group G, $G \geqq K \geqq H$ and K/H is torsion, then K is also a regular subgroup of G.*

PROOF. For any $k \in K$, we have $T_K(k)$ (type of k in K) $\leqq T_G(k)$. Since K/H is torsion, there is an integer $n \neq 0$ such that $nk \in H$. From the fact that H is regular, we have $T_G(k) = T_G(nk) = T_H(nk) \leqq T_K(nk) = T_K(k) \leqq T_G(k)$. Hence we get $T_K(k) = T_G(k)$.

From this lemma we see that any subgroup K of a torsion free group with G/K of bounded order is a regular subgroup of G.

THEOREM 1. *$G = \sum_{\lambda \in \Lambda} G_\lambda$ (direct sum) where Λ is a well ordered set with least element 0, G_λ has rank 1 for $\lambda > 0$ and $T(x) \geqq T(y)$, whenever $x \in G_\lambda$, $y \in G_\mu$ with $\lambda < \mu$. Then for any regular subgroup H of G we have $H = (H \cap G_0) \oplus \sum_{\lambda > 0} H_\lambda$ where $H_\lambda \cong G_\lambda$ or $H_\lambda = 0$.*

PROOF. Define $G^a = \sum_{\lambda < a} G_\lambda$ and $H^a = H \cap G^a$. Then $H^{a+1}/H^a = H^{a+1}/(H^{a+1} \cap G^a) \cong \{G^a, H^{a+1}\}/G^a \leqq G^{a+1}/G^a \cong G_a$. It is clear that H^a is pure in H for each $a \in \Lambda$. If $H^{a+1} \neq H^a$ and $a > 0$, for any $x \in H^{a+1}$ and $x \notin H^a$ we have $T_H(x) = T_G(x) = T(G_a)$. Hence $T(G_a) \leqq T(H^{a+1}/H^a) \leqq T(G_a) = T(x)$ for $a > 0$. By

BAER'S LEMMA. *Let S be a pure subgroup of the torsion free group G such that (1) G/S is of rank 1; (2) G/S is of type a and (3) every element of G not in S is again of type a. Then S is a direct summand of G.*

[1, Lemma 46.3, p. 163], H^a is a direct summand of H^{a+1} and $H^{a+1} = H^a \oplus H_a$, where H_a either vanishes or is $\cong G_a$ for $a > 0$ and $H_0 = H \cap G_0$. With a simple transfinite induction we are led to the result H^{a+1} is the direct sum of all H_λ with $\lambda \leqq a$, and $H = \sum_{\lambda \in \Lambda} H_\lambda = (H \cap G_0) \oplus \sum_{\lambda > 0} H_\lambda$.

COROLLARY 1. *Let G satisfy the conditions of Theorem 1 and $nG \leqq H \leqq G$ for some $n \neq 0$. Then $H = (H \cap G_0) \oplus \sum_{\lambda > 0} H_\lambda$, where $H_\lambda \cong G_\lambda$ for $\lambda > 0$ and $nG_0 \leqq H \cap G_0 \leqq G_0$.*

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This corollary follows immediately from the fact that $H^{\alpha+1} \neq H^{\alpha}$.

COROLLARY 2. *Let H be torsion free and $nH \leqq G \leqq H$ for some $n \neq 0$, where G is as in Theorem 1. Then $H = \sum_{\lambda \in \Lambda} H_\lambda$ with $H_\lambda \cong G_\lambda$ for $\lambda > 0$ and $nH_0 \leqq G_0 \leqq H_0$. Moreover, if G_0 is pure in H, H_0 can be taken to be G_0.*

PROOF. Since $nG \leqq nH \leqq G$, Corollary 1 gives $nH = (nH \cap G_0) \oplus \sum_{\lambda > 0} K_\lambda$ with $K_\lambda \cong G_\lambda$ for each $\lambda > 0$. The map $x \to (1/n)x$ is an isomorphism of nH onto H so that $H = (1/n)(nH \cap G_0) \oplus \sum_{\lambda > 0} H_\lambda$ with $H_\lambda = (1/n)K_\lambda \cong G_\lambda$ for $\lambda > 0$ and, taking $H_0 = (1/n)(nH \cap G_0)$, with $nH_0 \leqq G_0 \leqq H_0$. Finally if G_0 is pure in H, then $nH \cap G_0 = nG_0$ so that $H_0 = (1/n)(nH \cap G_0) = G_0$. Corollaries 1 and 2 give a generalization of a theorem of R. A. Beaumont and R. S. Pierce [**2**, Theorem 9.5 and Corollary 9.6]. When G_0 is taken to be zero, Theorem 1 and its corollaries apply to completely decomposable groups.

COROLLARY 3. *Let G be a torsion free group, $nG \leqq H \leqq G$ and H is a completely decomposable homogeneous group (every element is of the same type). Then G is isomorphic to H.*

THEOREM 2. *Let G be a finite rank torsion free group with types in a chain. Then G is completely decomposable if and only if for any regular subgroup H, with $r(H)$ (rank of H) $= r(G)$, G/H is finite.*

PROOF. Assume $G = \sum_{i=1}^{n} G_i$, $r(G_i) = 1$ and H is as stated in the theorem. Then by regularity, $H \cap G_i$ are of finite index in G_i. Since H contains $\sum_{i=1}^{n} (H \cap G_i)$ which is of finite index in G, we see that the condition is necessary. In order to prove sufficiency, we choose a special set of n independent elements $h_1, h_2, \cdots, h_n$ in the following manner. As G is of finite rank and with types in a chain, there must be an element $h_1 \neq 0$ with maximal type in G otherwise we shall be led to the contradiction that G is not of finite rank. Suppose we have selected $h_1, \cdots, h_r$ which are independent and K_r to be the set of elements which are independent to $h_1, \cdots, h_r$. In K_r we choose h_{r+1} with maximal type among the elements in K_r. With this selection we have a maximal independent set $\{h_1, \cdots, h_n\}$ and $H = \sum_{i=1}^{n} \langle h_i \rangle_*$ ($\langle x \rangle_*$ is the pure subgroup generated by x) is regular. For if H is not regular, there exists $h = \sum_{i=1}^{m} x_i$ ($x_i \in \langle h_i \rangle_*$ and $x_m \neq 0$) such that $T_G(x_m) = T_H(h) < T_G(h)$. Then if $h_1, \cdots, h_{m-1}, h$ is independent and $T(h) > T(h_m)$ we get a contradiction about the choice of h_i. Hence $H = \sum_{i=1}^{n} \langle h_i \rangle_*$ is regular and G/H finite. By Corollary 2, we have that G is completely decomposable. When we make the sharper condition that G is homogeneous in Theorem 2, we get a theorem of Baer [1, Theorem 48.1, p. 173].

References

1. L. Fuchs, *Abelian groups*, Publ. House of the Hungarian Academy of Sciences, Budapest, 1958.

2. R. A. Beaumont and R. S. Pierce, *Torsion-free rings*, Illinois J. Math. **5** (1961), 61–98.

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ISOMETRIC ISOMORPHISMS OF MEASURE ALGEBRAS

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1. **Introduction.** We shall prove that two locally compact topological groups with isometrically isomorphic measure algebras are themselves isomorphic. This is similar to a result of Wendel (see [2]) proved for group algebras. We first characterize the absolutely continuous measures on a group in terms of the metric and algebra properties of the measure algebra. It then follows that if two locally compact groups have isometrically isomorphic measure algebras then they have isometrically isomorphic group algebras, and so our result follows from Wendel's theorem.

2. **Singularity in terms of norm.** Throughout this section X will denote a locally compact Hausdorff space. The set $M(X)$ of bounded complex Radon measures on X forms a Banach space (see [1, p. 57]).

Lemma 1. *If α, β are two complex numbers then*

$$|\alpha+\beta| + |\alpha-\beta| \leqq 2|\alpha| + 2|\beta|,$$

equality occurring if and only if $\alpha=0$ or $\beta=0$.

Proof. This result follows from the elementary inequalities $|\alpha\pm\beta| \leqq |\alpha|+|\beta|$ by considering the occasions in which these are both equalities.

Theorem 1. *Let $\mu_1, \mu_2 \in M(X)$. Then μ_1 and μ_2 are mutually singular if and only if*

$$\|\mu_1+\mu_2\| + \|\mu_1-\mu_2\| = 2\|\mu_1\| + 2\|\mu_2\|. \tag{1}$$

Proof. Put $\nu = |\mu_1| + |\mu_2|$. Both μ_1 and μ_2 are absolutely continuous with respect to ν, and so there are ν-integrable functions f_1, f_2 on X, such that $\mu_i(E) = (f_i.\nu)(E) = \int_E f_i(t)d\nu(t)$, $i=1, 2$, for each Borel set E in X. We have

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