SIMILARITY TRANSFORMATIONS OF HYPERSURFACES

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It is the purpose of this note to extend C. S. Hsü's theorem [4] to \( n \) dimensions, \( n \geq 2 \). In order to do that we make use of the computations of [3, pp. 89–90].

Notations. \( M_1, M_2 \) are \( n \)-dimensional closed orientable Riemannian manifolds of class \( C^3 \), imbedded in \( n + 1 \) dimensional Euclidean space. \( \Delta_2 \) is the second differential operator of Beltrami. \( \Delta_1 \) is the operator defined by

\[
\Delta_1 \sigma = g^{ij} \frac{\partial \sigma}{\partial x^i} \frac{\partial \sigma}{\partial x^j}.
\]

As usual, \( g_{ij} \) is the positive-definite metric tensor of \( M_1 \) and \( (g^{ij}) = (g_{ij})^{-1} \), \( R = g^{ij} R_{ij} \), where \( R_{ij} \) is the Ricci tensor. We also use repeated index for summation. We denote corresponding elements of \( M_2 \) by attaching accents.

Lemma [5, p. 30]. In a compact Riemannian manifold with positive-definite metric, if a function \( \sigma \) satisfies

\[
\Delta_2 \sigma \geq 0
\]

everywhere in the manifold, then \( \sigma \) is a constant.

Theorem. Given \( M_1, M_2 \) with positive \( R, R' \), respectively, and a diffeomorphism \( h: M_1 \rightarrow M_2 \) which preserves RI. (I is the first fundamental form.) Then \( h \) is a similarity.

Proof. It is sufficient to show that \( h \) followed by a homothetic transformation is a rigid motion. We first show that \( I'/I \) is a constant. Let \( I'/I = R/R' = e^{2\varepsilon} \). Then \( g'_{ij} = e^{2\varepsilon} g_{ij} \). By [3, pp. 89–90], we have

\[
(R + 2(n - 1) \Delta_2 \sigma + (n - 1)(n - 2) \Delta_1 \sigma) = R' e^{2\varepsilon}.
\]

Making use of the hypothesis, we obtain

\[
\Delta_2 \sigma = - \left\{ \frac{n - 2}{2} \right\} \Delta_1 \sigma.
\]

Since \( \Delta_1 \sigma \geq 0 \),

\[
- \Delta_2 \sigma = \Delta_2 (-\sigma) = \left\{ \frac{n - 2}{2} \right\} \Delta_1 \sigma \geq 0.
\]

Received by the editors January 10, 1963.

1 The author is indebted to C. S. Hsü for his suggestions.
By the lemma, $\sigma$ is a constant. Consequently, $I'/I$ is also a constant. Now we divide the proof into two different cases.

**Case 1.** For $n = 2$, the fact that $h$ followed by a homothetic transformation of a proportionality constant $(R'/R)^{-1/2}$ is the desired rigid motion follows from Cohn-Vossen's theorem.

**Case 2.** For $n > 2$, the above fact follows from a different argument (cf. [1, pp. 26–27]). It says that isometric hypersurfaces in Euclidean space of dimension greater than three are congruent or symmetric.

**References**