ON THE EXTENSION OF OPERATORS WITH RANGE
IN A C(K) SPACE

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1. Introduction. Let $K$ be a compact Hausdorff space and let $C(K)$ be the space of all the continuous real-valued functions on $K$ with the supremum norm. The purpose of this note is to prove the following

THEOREM. For $X = C(K)$ the following four statements are equivalent.

(i) For every two Banach spaces $Z \supset Y$ with $\dim(Z/Y) = 1$ and every operator $T$ from $Y$ into $X$ with a separable range there is an extension $\tilde{T}$ of $T$ from $Z$ into $X$ with $\|\tilde{T}\| = \|T\|$.

(ii) For every two Banach spaces $Z \supset Y$ with $\dim Y = 2$, $\dim Z = 3$ and every operator $T$ from $Y$ into $X$ there is an extension $\tilde{T}$ of $T$ from $Z$ into $X$ with $\|\tilde{T}\| = \|T\|$.

(iii) There is a $\lambda < 2$ such that for every two Banach spaces $Z \supset Y$ with $\dim(Z/Y) = 1$ and every operator $T$ from $Y$ into $X$ with a separable range there is an extension $\tilde{T}$ of $T$ from $Z$ into $X$ with $\|\tilde{T}\| \leq \lambda \|T\|$.

(iv) $K$ is an $F$-space in the terminology of Gillman and Jerison [4, p. 208]. That is, for every $f \in C(K)$ there is a $g \in C(K)$ such that $f(k) > 0$ implies $g(k) \geq 1$ and $f(k) < 0$ implies $g(k) \leq -1$.

Assuming the continuum hypothesis the following statement is also equivalent to the preceding ones.

(v) For every two Banach spaces $Z \supset Y$ and every operator $T$ from $Y$ into $X$ with a separable range there is an extension $\tilde{T}$ of $T$ from $Z$ into $X$ with $\|\tilde{T}\| = \|T\|$.

The equivalence of (i) and (iv) is essentially due to Aronszajn and Panitchpakdi [2].

Our main tool in the proof of the theorem will be the equivalence of each of (i), (ii) and (iii) with intersection properties of the cells in $X$. The connection between extension and intersection properties was first observed by Nachbin in [13] and then used in more general situations in [2], [7] and [11].

We consider only Banach spaces over the reals. By “operator” we always mean “bounded linear operator.” The cell $\{x; \|x - x_0\| \leq r\}$ is denoted by $S(x_0, r)$.
2. Proof of the theorem. From the proof of Theorem 1.8 in [11] it follows that a Banach space $X$ satisfies (i) if and only if for every collection of mutually intersecting cells in $X$, whose centers generate a separable subspace, there is a point in $X$ common to all the cells in the collection. Aronszajn and Panitchpakdi [2, §5, Theorem 2] showed that a $C(K)$ space has this intersection property if and only if $K$ satisfies a certain topological condition which, as remarked by Henriksen [8], is equivalent to (iv). Hence (i)$\iff$(iv).

From the proof of Theorem 1.8 in [11] it follows also that a Banach space $X$ satisfies (ii) if and only if for every collection of mutually intersecting cells in $X$, whose centers generate a 2-dimensional subspace, there is a point in $X$ common to all the cells in the collection. This fact is used in the following lemma, which is the essential step in the proof that (ii) implies (iv).

**Lemma.** Suppose $X = C(K)$ satisfies (ii) and let $f \in C(K)$. Let $\alpha_n, \beta_n, \gamma_n, \delta_n$ be sequences of real numbers tending to 0 and satisfying

$$\alpha_{n+1} < \beta_n < \alpha_n, \quad \gamma_n < \delta_n < \gamma_{n+1}, \quad n = 1, 2, \ldots.$$ 

Then there is a $g \in C(K)$ such that for every $n$

$$\beta_n \leq f(k) \leq \alpha_n \Rightarrow g(k) \geq 1,$$

$$\gamma_n \leq f(k) \leq \delta_n \Rightarrow g(k) \leq -1.$$ 

**Proof.** Let $\tau_{a,b,c,d}(t)$, where $a < b < c < d$, denote the trapezoidal function equal to 0 for $t \leq a$ and $t \geq d$, equal to 1 for $t \in [b, c]$ and linear in $[a, b]$ and $[c, d]$. Put

$$
\tau_{2n-1}(t) = \tau_{\alpha_{n+1}, \beta_n, \gamma_n, \delta_{n-1}}(t), \quad \tau_{2n}(t) = \tau_{\beta_{n-1}, \gamma_n, \gamma_n, \delta_{n+1}}(t),
$$

for $n = 1, 2, \ldots$ (where $\beta_0$ is any number $> \alpha_1$ and $\delta_0 < \gamma_1$). Clearly

$$
\sum_{n=1}^{\infty} \tau_n(t) = 1, \quad t \in [\gamma_1, \alpha_1], \ t \neq 0.
$$

Hence if $\lambda_n \to \lambda$ the function equal to $\sum \lambda_n \tau_n(t)$ for $t \neq 0$ and equal to $\lambda$ if $t = 0$ is continuous. In particular

$$u(t) = \sum_{n=1}^{\infty} \sin n^{-1} \tau_n(t), \quad t \neq 0, \ u(0) = 0,$$

$$v(t) = \sum_{n=1}^{\infty} \cos n^{-1} \tau_n(t), \quad t \neq 0, \ v(0) = 1,$$

are continuous. Let $m$ be any integer and let $\lambda_m$ be defined by
Consider the function
\[ \phi_m(t) = (-1)^m \lambda_m (\sin m-1 \cdot u(t) + \cos m-1 \cdot v(t)). \]
For \( m = 2n - 1 \) we have
\[ \phi_{2n-1}(t) = \max_{-\alpha_n \leq t \leq \alpha_n} \phi_{2n-1}(s) = \lambda_{2n-1} \quad \text{for } \beta_n \leq t \leq \alpha_n, \]
and also, by (1) and (2),
\[ \phi_{2n-1}(t) \leq \lambda_{2n-1} - 2 \quad \text{for } t \leq 0. \]
Similarly for \( m = 2n \)
\[ \phi_{2n}(t) = \max_{-\delta_n \leq t \leq \delta_n} \phi_{2n}(s) = -\lambda_{2n} \quad \text{for } \gamma_n \leq t \leq \delta_n, \]
\[ \phi_{2n}(t) \leq -(\lambda_{2n} - 2) \quad \text{for } t \geq 0. \]
By (3), (4), (5) and (6)
\[ \phi_m(t) - \phi_p(t) \leq \lambda_m + \lambda_p - 2 \quad \text{for every } m, p \text{ and } t. \]
Put \( \psi_m(k) = \phi_m(f(k)), \) \( m = 1, 2, \ldots \). Each \( \psi_m \) belongs to the 2-dimensional subspace of \( C(K) \) spanned by \( u(f(k)) \) and \( v(f(k)) \). By (7) the cells \( S(\psi_m, \lambda_m - 1) \) are mutually intersecting and hence by our assumption on \( X \) there is a \( g \in X \) with \( \|g - \psi_m\| \leq \lambda_m - 1, \) \( m = 1, 2, \ldots \). By (3) and (5)
\[ \beta_n \leq f(k) \leq \alpha_n \Rightarrow \psi_{2n-1}(k) = \lambda_{2n-1} \Rightarrow g(k) \geq 1, \]
\[ \gamma_n \leq f(k) \leq \delta_n \Rightarrow \psi_{2n}(k) = -\lambda_{2n} \Rightarrow g(k) \leq -1, \]
and this concludes the proof of the lemma.

We show now that (ii)\( \Rightarrow \) (iv). Let \( C(K) \) satisfy (ii) and let \( f \in C(K) \) with \( \|f\| = 1 \). By the lemma there are \( g', g'' \in C(K) \) such that for \( n = 1, 2, \ldots \)
\[ -1/(2n - 1) \leq f(k) \leq -1/2n \Rightarrow g'(k) \leq -1, \quad g''(k) \leq -1, \]
\[ 1/(2n + 1) \leq f(k) \leq 1/2n \Rightarrow g'(k) \geq 1, \]
\[ 1/2n \leq f(k) \leq 1/(2n - 1) \Rightarrow g''(k) \geq 1. \]
Hence \( g_1 = \max (g', g'') \) satisfies
\[ f(k) > 0 \Rightarrow g_1(k) \geq 1, \quad -1/(2n - 1) \leq f(k) \leq -1/2n \Rightarrow g_1(k) \leq -1, \]
\[ n = 1, 2, \ldots. \]
In a similar manner it follows that there is a $g_2 \in C(K)$ satisfying
\[
  f(k) > 0 \Rightarrow g_2(k) \geq 1, \quad -1/2n \leq f(k) \leq -1/(2n + 1) \Rightarrow g_2(k) \leq -1, \\
  n = 1, 2, \ldots.
\]

The function $g = \min(g_1, g_2)$ has the property required in (iv). Hence (ii) $\Rightarrow$ (iv).

A Banach space $X$ has property (iii) with a certain $\lambda$ if and only if for every collection of mutually intersecting cells $\{S(x_\alpha, r_\alpha)\}_{\alpha \in A}$ in $X$ such that the set $\{x_\alpha\}_{\alpha \in A}$ generates a separable subspace, there is an $x \in X$ satisfying $\|x - x_\alpha\| \leq \lambda r_\alpha$ for every $\alpha \in A$ (cf. the proofs of [7, Theorem 1; 11, Theorem 1.8]). Let $C(K)$ satisfy (iii) and let $f \in C(K)$. Further let $r_\alpha(t)$ be equal to $1$ for $t \in [1/2, 1/2]$, equal to $-1$ for $t \leq -1/2$ and linear in $[-1/2, 1/2]$. The cells $\{S(r_\alpha(f), 1/2)\}_{\alpha = 1}$ are mutually intersecting. Let $g \in \cap_{\alpha} S(r_\alpha(f), \lambda/2)$. Since $X_2$ it follows that
\[
  f(k) > 0 \Rightarrow g(k) \geq 1 - \lambda/2, \quad f(k) < 0 \Rightarrow g(k) \leq -1 + \lambda/2.
\]

Hence $K$ satisfies (iv).

Since the implications (i) $\Rightarrow$ (ii) and (i) $\Rightarrow$ (iii) are trivial we have already shown that the statements (i)-(iv) are equivalent.

To conclude the proof of the theorem we show that, assuming the continuum hypothesis, (i) $\Rightarrow$ (v). This part of the proof is valid for general Banach spaces $X$. Let $X$ satisfy (i), let $Z \supset Y$ and let $T$ be an operator from $Y$ into $X$ with a separable range. The closed subspace $Y_0$ of $X$ generated by $TY$ is separable. Hence $Y_0$ is isometric to a subspace of $m$. Since the operator $T$ from $Y$ into $Y_0$ has a norm preserving extension from $Z$ into $m$ [3, p. 94], we have only to show that the identity map from $Y_0$ into $X$ has a norm preserving extension from $m$ into $X$. Hence we may assume that $Y$ is separable and $Z = m$. By the continuum hypothesis the cardinality of $m$ is $\aleph_1$. Let $\{z_\alpha\}_{\alpha \in \Omega}$ be a well ordering of the points of $m$, where $\Omega$ is the set of all ordinals smaller than the first uncountable one. For every $\alpha$ let $Z_\alpha$ be the closed subspace of $m$ spanned by $Y$ and $\{z_\beta\}_{\beta < \alpha}$. $Z_\alpha$ is separable for every $\alpha$. We now construct inductively for every $\alpha \in \Omega$ an operator $T_\alpha$ from $Z_\alpha$ into $X$ such that $T_\alpha = T_0 = T_\alpha$, and the restriction of $T_\alpha$ to $Z_\beta$ ($\beta < \alpha$) is equal to $T_\beta$. For $\alpha = \alpha' + 1$ let $T_\alpha$ be any norm preserving extension of $T_{\alpha'}$ to $Z_\alpha$. Such an extension exists by (i) ($\dim(Z_\alpha/Z_{\alpha'})$ is either 0 or 1). If $\alpha$ is a limiting ordinal we define $T_\alpha$ first on $\bigcup_{\beta < \alpha} Z_\beta$ by $T_\beta z = T_\beta z$ if $z \in Z_\beta$, $\beta < \alpha$ (the definition is easily seen to be independent of the choice of $\beta$), and extend it by continuity to $Z_\alpha$, which is the closure of $\bigcup_{\beta < \alpha} Z_\beta$. Having constructed the $T_\alpha$, $\alpha \in \Omega$, we define $\hat{T}$ on $m(= \bigcup_{\alpha \in \Omega} Z_\alpha)$ by $\hat{T}z = T_\alpha z$ if $z \in Z_\alpha$. $\hat{T}$ is clearly
a norm preserving extension of $T$ from $m$ into $X$. This concludes the proof of the theorem.

3. **Remarks.**

1. If we do not restrict ourselves to $C(K)$ spaces $X$, (i), (ii) and (iii) are no longer equivalent. $c_0$, for example, satisfies (ii) [12, Chapter 2, §6] but not (iii) (cf. Sobczyk [14]). Every finite-dimensional space satisfies (iii) [7, p. 198] but in general it does not satisfy (ii).

2. The fact that (i) and (iii) are equivalent for $C(K)$ spaces is similar to a result, due to Amir [1] and Isbell and Semadeni [9], that if a $C(K)$ space has a projection constant $\lambda<2$ then it has already the projection constant 1 (i.e., $K$ is extremally disconnected).

3. The requirement that the norm of the extension $\tilde{T}$ is exactly equal to that of $T$ is essential in (ii). Grothendieck [6] proved that every $C(K)$ space $X$ has the following extension property: For every two Banach spaces $Z \supseteq Y$ and every compact operator $T$ from $Y$ into $X$ there is, for every $\varepsilon>0$, a compact extension $\tilde{T}$ of $T$ from $Z$ into $X$ with $\|\tilde{T}\| \leq (1+\varepsilon)\|T\|$. (Cf. also [10], [12] for characterizations of the spaces $X$ having this extension property.)

4. In property (iii) we cannot take $\lambda=2$. Indeed, for every two Banach spaces $Z \supseteq Y$ with $\dim Z/Y=1$ and every operator $T$ from $Y$ into a $C(K)$ space $X$ there is an extension $\tilde{T}$ of $T$ from $Z$ into $X$ with $\|\tilde{T}\| \leq 2\|T\|$. As shown by Grünbaum [7] this assertion is equivalent to the following: For every collection of mutually intersecting cells $\{S(f_a, r_a)\}_{a \in A}$ in a $C(K)$ space $X$ there is an extension $\tilde{T}$ of $T$ from $Z$ into $X$ with $\|\tilde{T}\| \leq 2\|T\|$. As shown by Grünbaum [7] this assertion is equivalent to the following: For every collection of mutually intersecting cells $\{S(f_a, r_a)\}_{a \in A}$ in a $C(K)$ space $X$ there is a collection of cells $\{S(f_a, r_a)\}_{a \in A}$ such that $\cap_{a \in A} S(f_a, r_a) \cap 0$. That a $C(K)$ space has this intersection property follows immediately from the following two statements. (a) Let $X$ be a Banach space and let $\{S(x_a, r_a)\}_{a \in A}$ be a collection of mutually intersecting cells in $X$. Then for every $\varepsilon>0$, $\cap_{a \in A} S(x_a, 2r_a+\varepsilon) \cap 0$. (b) Let $X$ be a $C(K)$ space and let $\{S(f_a, r_a)\}_{a \in A}$ be a collection of cells in $X$. If for every $\varepsilon>0$, $\cap_{a \in A} S(f_a, r_a+\varepsilon) \cap 0$, then also $\cap_{a \in A} S(f_a, r_a) \cap 0$. Statement (a) was proved by Grünbaum [7, p. 198]. Actually he only asserts that (with the same notation as in (a)) $\cap_{a \in A} S(x_a, (2+\varepsilon)r_a) \cap 0$, but his proof shows that also (a) holds. Statement (b) follows from the following observation (contained implicitly in the proof of Theorem 2 in §5 of [2]): $\cap_{a \in A} S(f_a, r_a) \cap 0$ if and only if for every $k_0 \in K$ 

$$\lim_{k \rightarrow k_0} \sup_{a} (f_a(k) - r_a) \leq \lim_{k \rightarrow k_0} \inf_{a} (f_a(k) + r_a),$$

where $\lim_{k \rightarrow k_0} \sup_{a} \inf_{a}$ denotes the inf, over the neighborhoods $G$ of $k_0$, of $\lim_{k \rightarrow k_0} \sup_{a} \inf_{a}$ is defined analogously.

In the terminology of Grünbaum [7] this remark (together with a result of Amir [1]) means that the expansion constant of a $C(K)$
space is exact, being 1 for extremally disconnected $K$ and 2 in all other cases.

5. The range of the extension $\tilde{T}$ in property (v) will not in general be separable. For example, if $T$ is the identity map from a separable nonreflexive subspace $Y$ of $X$, into $X$, and if $Z = m (\supset Y)$, then every extension $\tilde{T}$ of $T$ from $Z$ into $X$ has a nonseparable range. This follows from a theorem of Grothendieck [5] which asserts that every bounded operator from $m$ into a separable space is weakly compact.

6. Let $m$ be an infinite cardinal and let (i-m), (iii-m) and (v-m) be the properties obtained from (i), (iii) and (v) by replacing the requirement that $T$ has a separable range by the requirement that the range of $T$ has a dense set of cardinality at most $m$. As in the proof of the theorem it can be shown that for $C(K)$ spaces (i-m) is equivalent to (iii-m) and that if we assume the generalized continuum hypothesis, (i-m) and (v-m) are equivalent for general Banach spaces $X$.

4. Appendix. We shall prove now that the Banach spaces $X = C(K)$ with $K$ an $F$ space (we assume always that $K$ is compact Hausdorff) have also the following extension property

(vi) For every two Banach spaces $Z$ and $Y$, with $Z \supset X$ and $Y$ separable, and every operator $T$ from $X$ into $Y$ there is a norm preserving extension of $T$ from $Z$ into $Y$.

This is the “from” extension property corresponding to the “into” property (v). We assume the continuum hypothesis. The assertion stated above is an immediate consequence of Proposition 2 below.

**Proposition 1.** Property (vi) is equivalent to each of the following two properties.

(vii) $X^*$ is an $L_1$ space, and for every two Banach spaces $Z$ and $Y$, with $Z \supset X$ and $Y$ separable, and every operator $T$ from $X$ into $Y$ there is an extension of $T$ from $Z$ into $Y$.

(viii) $X^*$ is an $L_1$ space, and every operator from $X$ into a separable Banach space is weakly compact.

**Proof.** (vi)$\Rightarrow$(vii) follows from $(8)\Rightarrow(2)$ in Theorem 1 of [10] (the assumption in (8) there that $X$ has the metric approximation property can be discarded. This was shown in our paper “On the extension of operators with a finite-dimensional range,” submitted to the Illinois J. Math.). (vii)$\Rightarrow$(viii) follows from the fact that, for every set $I$, every operator from $m(I)$ ($=$ the space of all bounded real-valued functions on $I$) to a separable space is weakly compact (Grothendieck
(vii) in the case $Z$ is an $m(I)$ space containing $X$ (clearly such a $Z$ always exists). (viii)$\Rightarrow$(vi) follows from (2)$\Rightarrow$(9) in
Theorem 1 of [10].

**Proposition 2.** *Property (v) implies (vi).*

**Proof.** Let $X$ satisfy (v). That $X^*$ is an $L_1$ space was proved by
Grothendieck [5] (cf. (3)$\Rightarrow$(2) in Theorem 1 of [10]). Let $Y$ be a
separable Banach space and let $T$ be an operator from $X$ into $Y$. We
show that $T$ is weakly compact. It is sufficient to show that the re-
striction of $T$ to every separable subspace of $X$ is weakly compact.
Let $X_0$ be a separable subspace of $X$. By (v) there is an operator $T_0$
from $m(\supset X_0)$ into $X$ whose restriction to $X_0$ is the identity. By the
result of Grothendieck [5], cited in the proof of Proposition 1, $TT_0$ is
weakly compact. It follows that the restriction of $TT_0$, and hence of
$T$, to $X_0$ is also weakly compact.

We have shown, in particular, that every $C(K)$ space with $K$ an
$F$ space has property (viii). This generalizes a result of Grothendieck
in [5], where this assertion was proved for extremally disconnected $K$.
Semadeni (in an unpublished note\(^a\)) remarked that the proof of
Grothendieck actually holds for every basically disconnected $K$ (cf.
[4, p. 22] for the definition of this notion). There exist connected $F$
spaces [4, p. 211] and thus our result solves problem 9 of [9].

Concluding, we remark that there are $C(K)$ spaces satisfying (vi)
without $K$ being an $F$ space. This follows from examples given in [1]
and [9] and the fact that if $C(K_1)$ is isomorphic to $C(K_2)$ (as a Banach
space) and if $C(K_1)$ satisfies (vii) then also $C(K_2)$ satisfies (vii) and
hence (vi). It would be interesting to have a characterization of all
spaces $K$ for which $C(K)$ satisfies (vi).

**References**

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\(^a\) Cf. also [9, Proposition 8].
1. Introduction. In the study of Cauchy problems of the form

\[
du/dt + Au = f; \quad u(\tau) = T
\]

(where for example: \( t \mapsto u(t) \in \mathcal{C}^k(H) \) on \( (\tau, b] \); \( t \mapsto u(t) \in \mathcal{C}^0(D(A)) \) on \( [\tau, b] \); \( H \) is a Hilbert space; \( -A \) is a closed (unbounded) operator, infinitesimal generator of a strongly continuous semi-group; \( \mathcal{C}^k(H) \) is the space of \( k \)-times continuously differentiable functions of \( t \) with values in \( H \); the domain of \( A, D(A) \), has the graph topology; and \( f, T \) are suitable), the solution takes the appearance

\[
u(t) = G(t, \tau)u(\tau) + \int_\tau^t G(t, \xi)f(\xi)d\xi.
\]

Formally the Green's operator \( G(t, \xi) \) may be written \( G(t, \xi) = \exp[-A(t-\xi)] \) (for general results in this direction see for example [1; 2; 3]). In this article we propose to study representations related to (1.2) for solutions of general operational differential equations \( Su = f \) (the operators need not be differential operators of course but therein lies the motivation, see [4; 5]; cf. also the papers [3; 6; 7; 8; 9; 10]).

2. Basic framework. Let \( H \) be a Hilbert space and \( (S_0, S_0') \) a formally adjoint pair of closed densely defined operators in the sense...