# GÖDEL NUMBERINGS VERSUS FRIEDBERG NUMBERINGS

MARIAN BOYKAN POUR-EL[1]

In [3], Rogers discussed the concept of *Gödel numbering*. He defined a *semi-effective numbering*, constructed a semi-lattice of equivalence classes of semi-effective numberings, and showed that all Gödel numberings belong to the unique maximal element of this semi-lattice.

In [1], Friedberg gave a recursive enumeration without repetition of the set of partial recursive functions of a single variable. Friedberg's numbering is clearly a semi-effective numbering which is not a Gödel numbering.[2]

QUESTION. How do Friedberg numberings (Definition 1 below) compare with Gödel numberings? More generally, where do Friedberg numberings fit into Rogers' semi-lattice?

DEFINITION 1. A Friedberg numbering $\tau$ is a semi-effective numbering such that

(1) $D_\tau = N$,

(2) $\tau_i \neq \tau_j$ if $i \neq j$.

SUMMARY OF RESULTS.[3]

I. If $\tau$ is a Friedberg numbering then $[\tau]$ is a minimal element of Rogers' semi-lattice. (Theorem 1.)

II. There exists two Friedberg numberings $\tau^1$ and $\tau^2$ such that $[\tau^1]$ and $[\tau^2]$ are incomparable. (Theorem 2.)

As a consequence of I and II we see that Rogers' semi-lattice is not a lattice (which answers a question raised in [3]).

THEOREM 1. *Let $\rho$ be a semi-effective numbering, $\tau$ a Friedberg numbering. If $\rho \leqq \tau$, then $[\rho] = [\tau]$.*

PROOF. Since $\rho \leqq \tau$, there is a recursive function $g$, mapping $D_\rho$ onto $D_\tau$ (which equals $N$) such that $\rho_i = \tau_{g(i)}$ on $D_\rho$. Define $h$ by

$$h(i) = \mu y[g(y) = i \cdot y \in D_\rho]$$

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[2] For example Friedberg's $\tau$ does *not* have the property that given any recursively enumerable sequence $f_1, f_2, \cdots$ of partial recursive functions there is a general recursive function $h$ such that $\tau_{h(i)} = f_i$. Furthermore, $\tau$ does *not* satisfy the recursion theorem relative to $\tau$. For there exists a recursive function $h$ such that for all $n$, it is *not* the case that $\tau_{h(n)} = \tau_n$ (e.g. let $h(n) = 1 \dot{-} \operatorname{sgn}(n)$).

[3] We follow the notation of [3].

$h$ is a recursive function mapping $D_\tau$ into $D_\rho$. Thus $[\rho]=[\tau]$.

THEOREM 2. *There exists two Friedberg numberings $\rho$ and $\tau$ such that* $[\rho]\neq[\tau]$.

PROOF. Let $\rho$ be the Friedberg numbering constructed by Friedberg [1, corollary to Theorem 3] from the standard Gödel numbering $\pi$. Our construction of $\tau$ from $\rho$ is similar to Friedberg's construction of $\rho$ from $\pi$. However Case 1 of the priority scheme is essentially different.

Since $\rho$ is a semi-effective numbering and $\pi$ is the standard numbering, there is a recursive function $h$ such that

$$\rho_i = \pi_{h(i)}.$$

As in [1] 0 will always be unused and $\rho_0=\tau_0=u$.[4] Let $e_a$ be the value of $e$ which is being pursued at step $a$. Then the following three cases arise.

*Case* 1. $(\exists y)[y<a\cdot T(h(e_a), e_a, y)\cdot U(y)=x$. $x$ is a follower of $e_a]$.[5] Then release $x$.

*Case* 2. Case 1 does not hold and there exists an $x$ such that $\rho^a_{e_a}$ is identical with $\tau^{a-1}_x$. If this $x$ is free, then either $x\leqq e_a$ or $x$ has been previously displaced by $e_a$. If this $x$ is a follower, it is a follower of some $e\leqq e_a$. Otherwise $x=0$.

Then do nothing.

*Case* 3. Neither Case 1 nor Case 2 holds.

Then the following four acts, some of which may be vacuous, are carried out.

A. If $e_a$ has no follower, make the lowest unused $x$ other than 0 a follower of $e_a$.

B. Put into $\tau_x$ where $x$ is the follower of $e_a$ all the members of $\rho^a_{e_a}$. (Thus $\tau^a_x$ becomes identical to $\rho^a_{e_a}$.)

C. If for some $x'\neq x$, $\tau^{a-1}_{x'}$ is identical with $\rho^a_{e_a}$, then put into $\tau^{a-1}_{x'}$ *the smallest ordered pair* (in some standard enumeration of ordered pairs) which differs in its argument member from any ordered pair previously listed in any $\rho$ or $\tau$. Thus $x'$ is *displaced* by $e_a$.

D. If the $x'$ of act C is a follower of some $e'$, release it.

*The proof proceeds by means of four lemmas. The first three*, modifications of Lemmas 3, 4, 5 of [1], respectively, *show that $\tau$ is a Friedberg numbering. Lemma* 4 *shows that $\rho$ and $\tau$ are incomparable Friedberg numberings.* For the sake of completeness we give all details. (Note that our Lemmas 1 and 3 are actually simpler than the corresponding ones of [1].)

[4] $u$ is the everywhere undefined function.

[5] $T$ and $U$ are defined in [2].

LEMMA 1. *For every $e$ there is an $x$ such that $\tau_x = \rho_e$.*

PROOF. We first show that $e$ cannot have an infinite number of disloyal followers. For $e$ can lose a follower only by Case 1 or Case 3. Clearly $e$ can lose a follower at most once by Case 1. Furthermore, since $\rho$ is a Friedberg numbering, there exists a stage $a$ such that for every $e' < e$, either $\rho_{e'}^a$ has acquired an ordered pair never to be acquired by $\rho_e$ or $\rho_e^a$ has acquired an ordered pair never to be acquired by $\rho_{e'}$. In the first case, $\tau_x^{a'-1}$ can never be equal to $\rho_{e'}^{a'}$ when $x$ is a follower of $e$ and $a' \geqq a$. In the second case, if $e$ should acquire a new follower $x$, $\tau_x$ will acquire all members in $\rho_e$ and $\tau_x^{a'-1}$ cannot ever again be identical with $\rho_{e'}^{a'}$. Thus $e$ cannot have infinitely many disloyal followers.

Let $a_0$ be a step after which $e$ will never lose a follower. If Case 3 occurs infinitely often with $e_a = e$, then the first time it occurs after step $a_0$, $e$ either acquires a follower or it already has one. This follower (call it $x$) will be loyal. Hence $\tau_x = \rho_e$.

Suppose Case 3 occurs only finitely often with $e_a = e$. Then since Case 1 can occur at most once, Case 2 must occur infinitely often with $e_a = e$. Whenever Case 2 occurs there must be an $x$ such that $\tau_x^{a-1} = \rho_e^a$. Furthermore this $x$ is either a follower of some $e' \leqq e$ or it must be free or it must be 0. (Recall that if Case 2 occurs when $x$ is free, either $x \leqq e$ or $x$ has previously been displaced by $e$.) If $x = 0$, $\tau_0 = \rho_0 = u$. So assume $x > 0$ and $x$ is either free as required by Case 2 or a follower of some $e' \leqq e$. Our aim is to show that $x$ can take only a finite number of different values if $e = e_a$ and Case 2 occurs infinitely often.

Only a finite number of values of $x$ are $\leqq e$. By hypothesis, Case 3 occurs only finitely often with $e = e_a$. Hence only finitely many $x$'s are displaced by $e$. Hence there are only finitely many $x$'s such that $x$ is free.

Suppose $x$ is a follower of $e' \leqq e$. There are only finitely many followers of $\bar{e}$ for each $\bar{e}$ by the construction above. Hence there are only finitely many followers of $e' \leqq e$.

Thus only finitely many different numbers can serve as the $x$ of Case 2 with $e = e_a$. Since Case 2 arises infinitely often, there must be an $x$ such that $\tau_x^{a-1} = \rho_e^a$ infinitely often. Thus $\tau_x = \rho_e$.

Thus we see that 0 is the only value of $x$ which remains unused.

LEMMA 2. *If $x \neq x'$, $\tau_x$ and $\tau_{x'}$ are not the same finite function.*

PROOF. By construction (see Case 3).

LEMMA 3. *If $x \neq x'$ and $\tau_x$ and $\tau_{x'}$ have infinite domains, then $\tau_x$ and $\tau_{x'}$ are not the same function.*

PROOF. If $x>0$, $x$ must become a follower of some $e$. If $x$ is a disloyal follower, then after $x$ is released, $x$ can only acquire a new member when $x$ is displaced. But $x$ is never displaced by any $e \geqq x$ and only once by an $e<x$. Hence if $x$ is disloyal, $\tau_x$ has a finite domain. Thus if $\tau_x$ has an infinite domain, $x$ must be a loyal follower of some $e$. Similarly $x'$ must be a loyal follower of some $e'$. Thus $\tau_x=\rho_e$ and $\tau_{x'}=\rho_{e'}$. Since $x \neq x'$, $e \neq e'$. Furthermore, since $\rho$ is a Friedberg numbering $\rho_e \neq \rho_{e'}$ if $e \neq e'$. Thus $\tau_x \neq \tau_{x'}$.

LEMMA 4. *$\tau$ and $\rho$ are incomparable Friedberg numberings.*

PROOF. Suppose $\tau$ and $\rho$ are comparable. Then since $\tau$ and $\rho$ are Friedberg numberings there would exist a recursive permutation $g$ such that $\rho_i=\tau_{g(i)}$ for all $i$. Now $g=\rho_{\bar{e}}$ for some $\bar{e}$, and $\tau_{\bar{x}}=\tau_{g(\bar{e})}=\rho_{\bar{e}}$ for $\bar{x}=g(\bar{e})$. Since $g$ is a recursive permutation, it has an infinite domain. Hence $\bar{x}$ is a loyal follower of $\bar{e}$. (See the proof of Lemma 3.) Now let $a$ be the first step such that

(a) $e_a=\bar{e}$,

(b) $\bar{x}$ is a follower of $\bar{e}$,

(c) $(\exists y)(y<a \cdot T(h(\bar{e}), \bar{e}, y) \cdot U(y)=\bar{x})$.

Such a step must exist since $\pi_{h(\bar{e})}=\rho_{\bar{e}}=\tau_{\bar{x}}$ and hence $\pi_{h(\bar{e})}(\bar{e})=\rho_{\bar{e}}(\bar{e})=\tau_{\bar{x}}(\bar{e})=\bar{x}$. Thus there is a $y$ such that $T(h(\bar{e}), \bar{e}, y)$ and $U(y)=\bar{x}$. Thus Case 1 would be in order and $\bar{x}$ would be released—contradiction.

REMARK 1. In Lemma 4 we prove something stronger than mere incomparability—there is no partial recursive function $g$ such that $\rho_i=\tau_{g(i)}$ whenever $g$ is defined and $g$ is defined for all $i$ such that $\rho_i$ has an infinite domain.

COROLLARY 1. *Rogers' semi-lattice is not a lattice.*

REMARK 2. Let $P=[\rho]\cup[\tau]$. Then $P$ contains a semi-effective numbering in which each partial recursive function is repeated exactly twice. However, $P$ does not contain a Friedberg numbering.

REMARK 3. It is easy to see that for every $k$, $[\rho]$ contains a numbering $\rho^*$ such that $D_{\rho^*}=N$ and each partial recursive function is repeated exactly $k$ times. (More generally, it is easy to show that $[\rho]$ contains a numbering in which each partial recursive function is repeated infinitely many times.) *Hence there cannot be a* 1-1 *mapping $g$ of $D_\rho$ onto $D_{\rho^*}$ such that $\rho_i=\rho^*_{g(i)}$ even though $[\rho^*]=[\rho]$. This observation answers another question raised by Rogers* [3, p. 336].

OPEN PROBLEM. Let $Q$ be a minimal element of Rogers' semi-lattice. Does there exist a Friedberg numbering $\tau$ such that $\tau \in Q$?

In conclusion it may be remarked that both *Theorem* 2 and the

fact that a Friedberg numbering $\tau$ does not satisfy the recursion theorem relative to $\tau$ (footnote 2) may provide some limitation on the usefulness of replacing Gödel numberings by Friedberg numberings in certain technical investigations.

## Bibliography

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# THE CLOSED CONVEX HULL OF CERTAIN EXTREME POINTS

DWIGHT B. GOODNER

Let $H$ be a Hausdorff space and let $C(H)$ be the space of bounded continuous real-valued functions on $H$, $C(H)$ having the usual supremum norm. Also, let $S$ be the unit ball of $C(H)$, let $E$ be the set of extreme points of $S$, and let $K$ be the set of characteristic functions of open-and-closed sets in $H$.

It is known [**1**, §3, p. 10] that when $H$ is compact, $S$ is the norm closed convex hull of $E$ if and only if $H$ is totally disconnected. It is known also [**4**, p. 103] that if $H$ is extremally disconnected, then $S$ is the norm closed convex hull of $E$. The purpose of this paper is to give a necessary and sufficient condition for $S$ to be the norm closed convex hull of $E$. The theorem for compact $H$ mentioned above is a special case of our result (cf. [**3**, p. 247]). We are indebted to the referee for observing that if $H$ is completely regular, our result follows at once from the compact case (cf. [**3**, p. 88]). Our condition is also necessary and sufficient for each of the subsets $E$ and $K$ to be fundamental (cf. [**2**, p. 35]) in $C(H)$.

If $f$ is a function in $C(H)$, we shall call the set

$$Z(f) = \{h: h \in H, f(h) = 0\}$$

the zero-set of the function $f$. We observe that if $r$ is a real number, then the sets $\{h: f(h) = r\}$, $\{h: f(h) \geqq r\}$, and $\{h: f(h) \leqq r\}$ are zero-

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