ON THE STRUCTURE OF THE GREEN'S OPERATOR

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1. Introduction. In the study of Cauchy problems of the form

\[ \frac{du}{dt} + Au = f; \quad u(t) = T \]

(where for example: \( t \to u(t) \in \mathcal{C}^k(H) \) on \((\tau, b] \); \( t \to u(t) \in \mathcal{C}^0(D(A)) \) on \([\tau, b] \); \( H \) is a Hilbert space; \( -A \) is a closed (unbounded) operator, infinitesimal generator of a strongly continuous semi-group; \( \mathcal{C}^k(H) \) is the space of \( k \)-times continuously differentiable functions of \( t \) with values in \( H \); the domain of \( A \), \( D(A) \), has the graph topology; and \( f, T \) are suitable), the solution takes the appearance

\[ u(t) = G(t, \tau)u(\tau) + \int_{\tau}^{t} G(t, \xi)f(\xi)d\xi. \]

Formally the Green's operator \( G(t, \xi) \) may be written \( G(t, \xi) = \exp[-A(t-\xi)] \) (for general results in this direction see for example \([1; 2; 3]\)). In this article we propose to study representations related to (1.2) for solutions of general operational differential equations \( Su = f \) (the operators need not be differential operators of course but therein lies the motivation, see \([4; 5]\); cf. also the papers \([3; 6; 7; 8; 9; 10]\)).

2. Basic framework. Let \( H \) be a Hilbert space and \( (S_0, S'_0) \) a formally adjoint pair of closed densely defined operators in the sense...
of Browder [7]. Define $S_1 = S_0^*$ (then $S_0 \subseteq S_1$) and let $H_0 = D(S_0)$, $H_1 = D(S_1)$, where $H_0$ and $H_1$ have the graph topology. Then $H_0 \subseteq H_0 \subseteq H$ (algebraically and topologically) and following [10] we set $H_1 = H_0 \oplus B$ where $B$ is the so-called Cauchy space or space of abstract boundary conditions (see [7; 9; 10]). The symbol $\oplus$ denotes here an orthogonal direct sum (topological); when we wish to speak of a not necessarily orthogonal direct sum (topological) of two closed complementary subspaces $M_1$ and $M_2$ of a Hilbert space $M$ we will write $M = M_1 + M_2$ (see here [11, p. 482]). It will be assumed throughout that $S_0$ is 1-1 with $S_0^{-1}$ continuous and that $S_1$ is onto $H$. (Such hypotheses are verified in many problems of interest; they imply (see [7]) that $S_0'$ has a closed range $\mathcal{R}(S_0')$ and that $(S_0, S_0')$ has a solvable realization operator $\hat{S}$; $R(S_0)$ is clearly closed also.) Now we will call any topological supplement of $H_0$ in $H_1$ a Cauchy space $\Gamma$ and write $H_1 = H_0 + \Gamma$ where in general $H_0$ and $\Gamma$ are not orthogonal. Clearly any such $\Gamma$ is isomorphic to $B$ (both are isomorphic to $H_1/H_0$). Then operators $\hat{S}$ such that $S_0 \subseteq \hat{S} \subseteq S_1$ are characterized by linear subspaces $\hat{\Gamma}$ of $\Gamma$; that is, $\hat{H} = D(\hat{S})$ is the set $\{u_1: u_1 \in H_1; ju_1 \in \hat{\Gamma} \subseteq \Gamma\}$ where $j: H_1 \to \Gamma$ is the (open) projection determined by $H_0$ and $\Gamma$. Then $\hat{H} = H_0 + \hat{\Gamma}$ and $\hat{H}$ would be given the graph topology. The following diagram will be useful in illustrating the subject (note ker $S_1$ is closed in $H$ or $H_1$)

\[
\begin{array}{ccccccccc}
0 & \to & \ker S_1 \\
\downarrow & & \downarrow & & \downarrow \\
0 & \to & H_0 & \to & H_1 & \overset{j}{\to} & \Gamma & \to & 0 \\
S_0 & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
& & \overset{i}{\to} & \overset{i}{\to} & \overset{i}{\to} & R(S_0) & \to & H & \to 0 \\
0 & \to & 0 & & & & & & .
\end{array}
\]

The horizontal and vertical sequences are exact (and split by the Banach theorem of homomorphisms). The continuous maps $i$ (injection), $S_1$, and $j$ (projection) may be thought of as morphisms in the category of Hilbert spaces. Note now that $H_0 + \ker S_1$ is closed and hence a topological direct sum since if $u_n$ is Cauchy in $H_0 + \ker S_1$ with $u_n = u_{0n} + u_{1n}$ then $S_0 u_{0n}$ converges which implies that $u_{0n}$ converges. The diagram (2.1) may be further expanded as follows (cf. [7]), defining $\hat{\Gamma}$ to be any topological supplement of $H_0 + \ker S_1$ in $H_1$. 

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\[ H_0 + \ker S_1 + \Gamma \xrightarrow{j} \{0\} + \Gamma_0 + \tilde{\Gamma} \]

\[(2.2) \]

\[ \downarrow S_1 \]

\[ R(S_0) + \{0\} + \tilde{\Gamma} \]

where \( \Gamma_0 = j(\ker S_1) \), \( \tilde{\Gamma} = S_1 \Gamma \), and in abuse of notation we identify \( \Gamma \) and \( j\tilde{\Gamma} \). It is clear that \( \tilde{\Gamma} \) is closed since \( S_1 \) is open and an isomorphism on \( \Gamma \); to see that \( \tilde{\Gamma} \cap R(S_0) = \{0\} \), suppose the contrary. Thus if \( h_0 \in H_0 \), \( h \in \tilde{\Gamma} \), and \( S_0 h_0 = S_1 h \), it follows that \( h_0 - h \in \ker S_1 \); since \( h_0 - h \in H_1 \) we must have \( h_0 - h = 0 \). Evidently \( H = R(S_0) + \tilde{\Gamma} \).

We define the Green’s operator to be the map \( \mathcal{G} : (j\mu_1, S_1 u_1) \rightarrow u_1 \) \( \Gamma \times H \rightarrow H_1 \) which recovers \( u_1 \) from a knowledge of \( j\mu_1 \) and \( S_1 u_1 \). It is seen from the diagrams that \( \mathcal{G} \) is well defined (if \( j\mu_1 = S_1 u_1 = 0 \) then \( u_1 \in \ker S_1 \cap H_0 = \{0\} \)). Moreover suppose \( j\mu_1 \rightarrow 0 \) in \( \Gamma \) and \( S_1 u_1 \rightarrow 0 \) in \( H \); then writing \( u_1 = u_0 + u \), \( u_0 \in H_0 \), \( u \in \Gamma \), we have \( u \rightarrow 0 \) in \( \Gamma \) and \( S_1 u_0 + S_0 u_0 \rightarrow 0 \) in \( H \). Hence \( u \rightarrow 0 \) in \( H \), \( S_1 u \rightarrow 0 \) in \( H \), and \( S_1 u + S_0 u_0 \rightarrow 0 \) in \( H \). This implies \( S_0 u_0 \rightarrow 0 \) in \( H \) and therefore \( u_0 \rightarrow 0 \) in \( H \) by the continuity of \( S_0^{-1} \). Thus finally \( u_1 \rightarrow 0 \) in \( H_1 \) and we have

**Proposition 1.** The map \( \mathcal{G} : \Gamma \times H \rightarrow H_1 \) defined by \( \mathcal{G}(j\mu_1, S_1 u_1) = u_1 \) is continuous.

It should be noted that \( \mathcal{G} \) is not a bilinear map in the usual sense and is defined only on the set \( G = \{(j\mu_1, S_1 u_1)\} \subset \Gamma \times H \).

**3. Decomposition of the Green's operator.** By the preceding it follows that if \( j\mu_1 = 0 \) (i.e. \( u_1 \in H_0 \)) then \( \mathcal{G}(0, S_1 u_1) \) defines a continuous map \( \mathcal{G}_2 : R(S_0) \rightarrow H_1 \). Clearly on \( R(S_0) \), \( \mathcal{G}_2 \) may be written as \( S_0^{-1} = \tilde{S}_0^{-1} \) where \( \tilde{S} \) is a solvable realization operator for \( (S_0, S_1) \); hence \( \mathcal{G}_2 \) may be extended (as \( \tilde{S}_0^{-1} \)) to a continuous map \( \mathcal{G}_2 : H \rightarrow H_0 + \tilde{\Gamma} \) (cf. [7]). On the other hand if \( u_1 \in \ker S_1 \), then \( \mathcal{G}(j\mu_1, 0) \) determines a continuous map \( \mathcal{G}_1 : \Gamma_0 \rightarrow H_1 \) (the identity) which may be extended to a continuous map (the identity) \( \mathcal{G}_1 : \Gamma_0 + \tilde{\Gamma} \rightarrow H_1 \). Then for \( u_1 \in H_0 + \ker S_1 \)

\[(3.1) \]

\[ u_1 = \mathcal{G}_1(j\mu_1) + \mathcal{G}_2(S_1 u_1), \]

whereas for \( u_1 \in \tilde{\Gamma} \) we must have

\[(3.2) \]

\[ u_1 = \mathcal{G}_1(j\mu_1) = \mathcal{G}_2(S_1 u_1). \]

Our interpretation of (1.2) is

\[(3.3) \]

\[ u_1 = \mathcal{G}_2(\rho S_1 u_1) + \mathcal{G}_1(j\mu_1), \]

where \( \rho : H \rightarrow R(S_0) \) is the projection, determined by \( R(S_0) \) and \( \tilde{\Gamma} \). Another formula for the solution similar to (3.3) is
\( u_1 = \mathcal{G}_2(S_1 u_1) + \mathcal{G}_1(\beta j u_1), \)

where \( \beta: \Gamma \to \Gamma_0 \) is the projection, determined by \( H_0 \) and \( \ker S_1 \). Note that the split \( H_0 + \ker S_1 \) is predetermined; however there is still liberty in choosing \( \Gamma \) and hence \( \hat{H} \).

We recall now the notion of a kernel for an operator \( T: \mathcal{C} \to \mathcal{C}_1 \) (see here for example [12; 13; 14]); we consider kernels in the sense of Aronszajn and will not attempt to treat here situations requiring the Schwartz kernel theorem (see [15]). Assuming \( \mathcal{C} \) and \( \mathcal{C}_1 \) are separable Hilbert spaces of equivalence classes of measurable functions over a regular measure space \( (X, \mu) \) (see [12]), then \( T \) has a kernel \( T(y, \cdot) \) if: (1) for all \( y \in X \), \( T(y, \cdot) \in \mathcal{C} \); (2) the map \( y \mapsto T(y, \cdot): X \to \mathcal{C} \) is measurable; (3) for all \( h \in D(T) \), \( (T h)(y) = (h, T(y, \cdot)) \) almost everywhere. If for example all functions in the range of a bounded operator \( T \) are continuous then following Theorem 4 of [12] it is seen that \( T \) has a kernel \( T(y, \cdot) \). This will often prevail in applications (cf. [17]).

Suppose now that \( S_1 \) and \( S_2 \) have kernels \( g_1(\xi, \cdot) \) and \( g_2(\xi, \cdot) \); \( g_1 \) and \( g_2 \) are considered as operators in \( \Gamma \) and \( H \) respectively. Then for example (3.3) may be written (see [16] for extensions of (1.2))

\[ (3.5) \quad u_1 = (\rho S_1 u_1, g_2(\xi, \cdot))_H + (j u_1, g_1(\xi, \cdot))_H. \]

We denote the adjoints of continuous maps \( T \) by \( T^* \) and those of unbounded maps \( T \) by \( T^* \). Then from (3.5), since \( g_1 \in \Gamma \)

\[ (3.6) \quad u_1 = (u_1, iS_1 \rho g_2(\xi, \cdot) + jg_1(\xi, \cdot)) u_1, \]

The following exact sequences indicate how the maps work:

\begin{align*}
(1) & \quad 0 \to H \to H_0 \xrightarrow{j} R(S_0) \to 0; \\
(2) & \quad 0 \to H \ominus R(S_0) \to H \xrightarrow{j\rho} H \ominus \hat{H} \to 0; \\
(3) & \quad 0 \to H_0 \to H_1 \xrightarrow{j} \Gamma \to 0; \\
(4) & \quad 0 \to H_1 \ominus \Gamma \to H_1 \xrightarrow{j} H_1 \ominus H_0 \to 0; \\
(5) & \quad 0 \to \ker S_1 \to H_1 \xrightarrow{S_1} H \to 0; \\
(6) & \quad 0 \to H \xrightarrow{iS_1} H_1 \ominus \ker S_1 \to 0
\end{align*}

(note also \( \iota S_1: R(S_0) \to H_1 \ominus (\Gamma + \ker S_1) \) and \( \iota S_1: \hat{H} \to H_1 \ominus (H_0 + \ker S_1) \)).

It is seen that certain problems arise because of the fact that even if \( H_1 = (H_0 + \ker S_1) \oplus \Gamma \) it is not true necessarily that \( H = R(S_0) \oplus \hat{H} \), where \( \hat{H} = S_1 \Gamma \). For example if we choose \( \hat{H} \) first, orthogonal to
$R(S_0)$, and define $\tilde{T} = S_0\tilde{H}$, then $\tilde{T}$ is orthogonal to $H_0 + \ker S_1$; however then $S_0\tilde{T} \neq \tilde{H}$ in general.

**Proposition 2.** Assume $\xi_1$ and $\xi_2$ have kernels as above; then $H_1$ has a reproducing kernel given by

$$h_1(t, \cdot) = \xi_1 T g_1(t, \cdot) + T j g_1(t, \cdot). \tag{3.7}$$

We may relate $\xi_1$ to our original operators as follows. Assume $v \in H$ and $\xi_1 v = w$; then for all $u \in H_1$ we have $(S_1 u, v)_H = (u, w)_H$. This means $(S_1 u, v - S_1 w)_H = (u, w)_H$. Therefore $v - S_1 w \in D(S_-)$ and since $S_- = S_0'$ it follows that $w = S_0' (v - S_1 w)$ (recall $H_1$ is dense in $H$). Thus $w$ appears as a solution of the equation $(v - S_1 w) = (S_0')^{-1} w$. We note that $g_1(t, \cdot)$ as defined is a reproducing kernel for $\Gamma$ and thus for $u_1 \in \Gamma$ there results $u_1 = (u_1, h_1)_H = (u_1, g_1)_H$. In general $g_1$ is the component of $h_1$ in $\Gamma$ when $H_1$ is written in the form $\Gamma \oplus (H_1 \oplus \Gamma)$. It should be observed that $H_0$ orthogonal to $\ker S_1$ in $H_1$ is impossible and this fact is closely connected with the development which we have given. A result similar to (3.7) can also be obtained using (3.4). By virtue of the above we may now write (3.7) in a form suitable for calculation.

$$T \rho g_2 = ((S_0')^{-1} + S_1)(h_1 - T j g_1). \tag{3.8}$$

This formula will not however entirely determine $g_2$ in terms of $h_1$ and $g_1$; it defines $g_2$ up to a term in $H \oplus R(S_0)$. However, this is sufficient and we have

**Proposition 3.** The component of $g_2$ in $R(S_0)$ is determined by (3.8) if $h_1$ and $g_1$ are known. If therefore $\tilde{H}$ is chosen orthogonal to $R(S_0)$ (with $\tilde{\Gamma} = \tilde{S}^{-1}\tilde{H}$), then $\xi_2(\rho S_1 u_1)$ is fully determined by (3.8).

On the other hand let $h_1$ be given; then $g_1$ is determined as the component of $h_1$ in $\Gamma$ when $H_1$ is decomposed as $H_1 = \Gamma \oplus (H_1 \oplus \Gamma)$. Thus if $J$ is the orthogonal projection $J: H_1 \to \Gamma$ then $g_1 = J h_1$. Define then the element $\rho g_2 = (S_0^{-1}(h_1 - T j g_1))$. This is well-defined if $h_1 = h_0 + g_1$, $h_0 \in H_1 \oplus \Gamma$, $g_1 \in \Gamma$, then $j h_1 = j g_1 = g_1 \in H_1 \oplus H_0$ and since $t_1$ is a projection $h_1 - j g_1 \in H_1 \oplus \Gamma$; thus $h_1 - j g_1 \in H_1 \oplus \ker S_1$ with $'S_1^{-1}(h_1 - g_1)$ well defined. Now since $h_1 - j g_1 \in H_1 \oplus \Gamma$ we have $'S_1^{-1}(h_1 - j g_1) \in R(S_0)$ and thus $'\rho g_2 \in R(S_0)$. Assuming now $H = R(S_0) \oplus \tilde{H}$ with $\tilde{\Gamma} = \tilde{S}^{-1}\tilde{H}$, it follows that $'\rho g_2$ defines an element $g_2 (= '\rho g_2)$ in $R(S_0)$ with

$$\rho S_1 u_1, g_2)_H = (S_1 u_1, '\rho g_2) = (S_1 u_1, 'S_1^{-1}(h_1 - j g_1)) = (u_1, h_1 - j g_1) = g_2(\rho S_1 u_1). \tag{3.9}$$

Hence $\xi_2$ has a kernel $g_2$ in $R(S_0)$ given by $'p^{-1} 'S_1^{-1}(h_1 - j g_1)$.  

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Proposition 4. Assume \( H_1 \) has a reproducing kernel \( h_1 \) and \( H = R(S_0) \oplus \bar{H} \). Then \( G_2 \) has a kernel in \( R(S_0) \) determined by (3.8).

Added in proof. The results of this paper are used in constructing abstract Green's operators in [16] for problems related to [5]. It is shown that \( \tilde{S} = S^* \) (notations of [5]) and formulas such as (3.8) and (1.2) can be studied in more detail.

Bibliography


Rutgers, The State University