ON TOPOLOGICALLY INDUCED GENERALIZED PROXIMITY RELATIONS

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1. **Introduction.** In this paper we examine one of the results in the theory of the proximity spaces of Efremovic [1]:

A set X with a binary relation "A close to B" (written $A\ \delta\ B$) is a proximity space if and only if there exists a compact Hausdorff space Y in which X can be topologically imbedded so that

(1.1) $\qquad A\ \delta\ B$ *in* X *if and only if* $\bar{A}$ *meets* $\bar{B}$ *in* Y

($\bar{A}$ denotes the closure of the set A) [2].

This proposition raises the question: Can we characterize the relations δ for which this result holds under weaker conditions on Y? In §4 we give an affirmative answer (Theorem 5.3) using rather mild restrictions on Y and on the imbedding of X in Y. This result is essentially a corollary to a fundamental theorem (Theorem 4.2).

2. **Symmetric generalized proximity spaces.** As in [3] we define a *symmetric generalized proximity space* (or *P_s-space*) to be an abstract set X with a binary operation "$A\ \delta\ B$" (a *P_s-relation*) on its power set satisfying the following axioms:

(P.1) $A\ \delta\ (B\cup C)$ implies that either $A\ \delta\ B$ or $A\ \delta\ C$.

(P.2) $A\ \delta\ B$ implies that A and B are nonvoid.

(P.3) If A meets B then $A\ \delta\ B$.

(P.4) $A\ \delta\ B$ and $b\ \delta\ C$ for all b in B implies that $A\ \delta\ C$.

(P.5) $A\ \delta\ B$ implies $B\ \delta\ A$.

We read the symbols "$A\ \delta\ B$" as "A is close to B"; and we say that "A is remote from B" (in symbols, "A not $\delta\ B$") if A is not close to B.

(2.1) The following facts are evident: (1) If $A\ \delta\ B$, $A\subset C$, and $B\subset D$, then $C\ \delta\ D$. (2) Define

$$A^{\delta} = \{x\in X: x\ \delta\ A\};$$

then in a P_s-space $(A^{\delta})\ \delta\ (B^{\delta})$ if and only if $A\ \delta\ B$.

(2.2) In [3] it is shown that there is a topology induced on every P_s-space $(X,\ \delta)$ by the closure operation $A\rightarrow A^{\delta}$. Moreover, this topology is *symmetric*: x in $\bar{y}$ implies y in $\bar{x}$ for all points $x,\ y\in X$. Clearly, every T_1 topological space is symmetric.

(2.3) THEOREM. *Given any symmetric topological space X define δ_0 by*:

Received by the editors October 13, 1962 and, in revised form, February 9, 1963.

(2.4) $A\ \delta_0\ B$ *if and only if* $\overline{A}$ *meets* $\overline{B}$.

Then δ_0 *is a* P_s*-relation and is compatible with the given topology*: $x\ \delta_0\ B$ *if and only if* $x \in \overline{B}$.

PROOF. We derive axioms (P.1) through (P.5) by use of the Kuratowski closure axioms [4]. Axioms (P.1), (P.2), (P.3) and (P.5) are trivial results of the closure axioms and (2.4). For (P.4), note that if for a point b and a set C we have $\bar{b} \cap \overline{C} \neq \varnothing$, then there exists a point c in C such that $c \in \bar{b}$. By symmetry then $b \in \bar{c} \subset \overline{C}$. Thus, if $\overline{A} \cap \overline{B} \neq \varnothing$ and $\bar{b} \cap \overline{C} \neq \varnothing$ for every b in B then $\overline{B} \subset \overline{C}$ and so $\overline{A} \cap \overline{C} \neq \varnothing$. It is now clear, from the above argument, that δ_0 is compatible with the given topology.

(2.5) THEOREM. *Given a* P_s*-space* (X, δ) *and* δ_0 *defined by* (2.4) *in terms of the topology induced by* δ *we have that* $A\ \delta_0\ B$ *implies that* $A\ \delta\ B$ *for all subsets* A *and* B *of* X. *Thus* δ_0 *is the smallest P-relation compatible with the topology in a symmetric topological space.*

PROOF. The demonstration follows directly from (2.3), (2.1) and (P.3).

3. **Clusters.** A *cluster* π from a P_s-space (X, δ) is a class of subsets of X satisfying:

(C.1) $A\ \delta\ B$ for all $A, B \in \pi$.

(C.2) $A \cup B \in \pi$ implies that either $A \in \pi$ or $B \in \pi$.

(C.3) If $B\ \delta\ A$ for every A in π, then $B \in \pi$.

Note that this is the same definition used by Leader [5] in introducing clusters for Efremovic proximity spaces.

(3.1) THEOREM. *For* x, *a point in a* P_s*-space* (X, δ), *the class* π_x *of all subsets of* X *which are close to* x *is a cluster from* X.

PROOF. We must show that π_x satisfies (C.1), (C.2) and (C.3). For (C.1) suppose $A, B \in \pi_x$. Then $x\ \delta\ A$ and $x\ \delta\ B$ so that, by (2.5), $A\ \delta\ B$. For (C.2) suppose $A \cup B \in \pi_x$. Then $x\ \delta\ (A \cup B)$ and, by (P.1), this means that either $x\ \delta\ A$ or $x\ \delta\ B$, that is, either $A \in \pi_x$ or $B \in \pi_x$. For (C.3) suppose that $A\ \delta\ C$ for every C in π_x. Since, by (P.3), $\{x\} \in \pi_x$, we have in particular, that $A\ \delta\ x$ or, $A \in \pi_x$.

(3.2) The following facts are easily established. (1) Any cluster π from a P_s-space (X, δ) is closed under the operation of supersets: if π is a cluster from X, $A \in \pi$, and $A \subset B$, then $B \in \pi$. (2) If $A \in \pi$, a cluster from X, and $a\ \delta\ B$ for every a in A, then $B \in \pi$. (3) If π and π' are clusters from X and π is a subclass of π', then $\pi = \pi'$. (4) If a point x belongs to a cluster π, then π is just the class π_x of all subsets A of X such that $A\ \delta\ x$. (5) Given a cluster π from a nonvoid P_s-space

(X, δ) and any subset A of X, then either $A \in \pi$ or $X - A \in \pi$. (6) Let π be a cluster from (X, δ). If A is a subset of X which meets every member of π, then $A \in \pi$.

4. **Extensions characterized by clusters.** We say that a subset X of a topological space Y is *regularly dense* in Y if and only if given U open in Y and p a point in U there exists a subset E of X with $p \in \overline{E} \subset U$, the closure being taken in Y.

(4.1) THEOREM. *If X is regularly dense in Y, then X is dense in Y. If Y is regular and X is dense in Y then X is regularly dense in Y.*

PROOF. Y is open in Y, hence for any point p in Y there exists a subset E of X such that $p \in \overline{E} \subset \overline{X} \subset Y$. Since this is true for any p in Y, we have $Y \subset \overline{X} \subset Y$.

For Y regular, $y \in Y$ and U an open set of Y containing y we have the existence of an open set V of Y containing y such that $\overline{V} \subset U$. Now $E = V \cap X$ is a subset of X and $\overline{E} = \text{Cl}(V \cap X) = \overline{V} \subset U$,[1] with the second equality following from the density of X in Y. Thus, $y \in \overline{E} \subset U$.

(4.2) THEOREM. *Given a set X and some binary relation δ on the power set of X, the following are equivalent:*

(I) *There exists a T_1 topological space Y and a mapping f of X into Y such that fX is regularly dense in Y and*

(4.3) $\quad$ *$A \;\delta\; B$ in X if and only if* $\text{Cl}(fA)$ *meets* $\text{Cl}(fB)$ *in Y.*

(II) *δ is a P_s-relation satisfying the additional axiom,*

(P.6) *Given $A \;\delta\; B$ in X there exists a cluster π to which both A and B belong.*

PROOF. Suppose that (I) holds and define δ by (4.3). (P.1), (P.2), (P.3) and (P.5) are trivial consequences of the properties of closure. For (P.4) suppose that $A \;\delta\; B$ and $b \;\delta\; C$ for all b in B. Then $\text{Cl}(fA) \cap \text{Cl}(fB) \neq \varnothing$ and $\text{Cl}(fb) \cap \text{Cl}(fC) \neq \varnothing$ for all b in B, which since Y is T_1, implies that $fb \in \text{Cl}(fC)$ for all b in B. Thus $fB \subset \text{Cl}(fC)$ or $\text{Cl}(fB) \subset \text{Cl}(fC)$ so that $\text{Cl}(fA) \cap \text{Cl}(fC) \neq \varnothing$ showing that $A \;\delta\; C$. For (P.6), since $\text{Cl}(fA) \cap \text{Cl}(fB) \neq \varnothing$, let $c \in \text{Cl}(fA) \cap \text{Cl}(fB)$ and define π to be the class of all subsets S of X such that $c \in \text{Cl}(fS)$. Clearly A and B are in π and in showing that π is a cluster the demonstrations of (C.1) and (C.2) are trivial. For (C.3) suppose that $\text{Cl}(fD) \cap \text{Cl}(fC) \neq \varnothing$ for every C in π but that $D \notin \pi$, i.e., $c \notin \text{Cl}(fD)$. Thus, $c \in Y - \text{Cl}(fD)$ and since FX is regularly dense in Y there exists a subset E of X such that $c \in \text{Cl}(fE) \subset Y - \text{Cl}(fD)$. That is, there

[1] Where Cl stands for closure.

exists an E in π such that $\mathrm{Cl}(fD)\cap\mathrm{Cl}(fE)=\varnothing$. This contradicts the hypothesis of (C.3). Thus (II) is satisfied.

For the converse suppose that (II) holds. Given x in X the class π_x of all subsets A of X such that $x\ \delta\ A$ is a cluster from X, by (3.1). Thus for any subset A of X, let $\mathfrak{A}$ be the set of all clusters π_a determined by the points a in A. Let $\bar{\mathfrak{A}}$ be the set of all clusters to which A belongs. By (P.3), $A\in\pi_a$ for each a in A and so $\mathfrak{A}\subset\bar{\mathfrak{A}}$. We will denote $\bar{\mathfrak{X}}$, the set of all clusters from X, by Y.

A subset A of X absorbs a subset β of Y if and only if A belongs to every cluster in β, that is, if and only if $\bar{\mathfrak{A}}$ contains β. For any subset β of Y we define the closure, $\mathrm{cl}(\beta)$, of β by

(4.4) $\pi\in\mathrm{cl}(\beta)$ if and only if every subset E of X which absorbs β is in π.

We next show that

(4.5) $\mathrm{cl}(\mathfrak{A})=\bar{\mathfrak{A}}$.

For if $\pi\in\mathrm{cl}(\mathfrak{A})$ then since A absorbs $\mathfrak{A}$, $A\in\pi$ so that $\pi\in\bar{\mathfrak{A}}$. On the other hand, if $\pi\in\bar{\mathfrak{A}}$ then $A\in\pi$. Now let P be in every π_a in $\mathfrak{A}$, i.e., $P\ \delta\ a$ for every a in A and hence $A\subset P^{\delta}$. Thus, by (3.2), (2), $P\in\pi$ so that $\pi\in\mathrm{cl}(\mathfrak{A})$.

We now show that the Kuratowski closure axioms are satisfied by the closure defined by (4.4).

(K.1) $\beta\subset\mathrm{cl}(\beta)$: This is trivial since if E absorbs β then $E\in\pi$ for every $\pi\in\beta$.

(K.2) $\mathrm{cl}(\varnothing)=\varnothing$: Suppose $\pi\in\mathrm{cl}(\varnothing)$. Since it is vacuously true that every subset of X absorbs $\varnothing$, we then have that every subset of X is in π. In particular, $\varnothing$ and X are in π. Thus, $\varnothing\ \delta\ X$, by (C.1), contradicting (P.2).

(K.3) $\mathrm{cl}(\mathrm{cl}(\beta))\subset\mathrm{cl}(\beta)$: Suppose $\pi\in\mathrm{cl}(\mathrm{cl}(\beta))$ and that E absorbs β. By (4.4), E absorbing β implies that E absorbs $\mathrm{cl}(\beta)$. Hence $E\in\pi$ showing that $\pi\in\mathrm{cl}(\beta)$.

(K.4) $\mathrm{cl}(\beta\cup\beta')=\mathrm{cl}(\beta)\cup\mathrm{cl}(\beta')$: Suppose that $\pi\in\mathrm{cl}(\beta\cup\beta')$ and that A absorbs β and A' absorbs β'. Then, by (3.2), (1), $A\cup A'$ absorbs $\beta\cup\beta'$ so that $A\cup A'\in\pi$. But, by (C.2), this means that either $A\in\pi$ or $A'\in\pi$, that is $\pi\in\mathrm{cl}(\beta)$ or $\pi\in\mathrm{cl}(\beta')$. Thus $\pi\in\mathrm{cl}(\beta)\cup\mathrm{cl}(\beta')$ and we have $\mathrm{cl}(\beta\cup\beta')\subset\mathrm{cl}(\beta)\cup\mathrm{cl}(\beta')$. On the other hand, $\pi\in\mathrm{cl}(\beta)\cup\mathrm{cl}(\beta')$ implies that either $\pi\in\mathrm{cl}(\beta)$ or $\pi\in\mathrm{cl}(\beta')$. Now if E absorbs $\beta\cup\beta'$, then E absorbs β and also absorbs β'. Hence, $E\in\pi$ showing that $\pi\in\mathrm{cl}(\beta\cup\beta')$ and (K.4) holds.

To show that the topology is T_1, suppose $\pi'\in\mathrm{cl}(\pi)$, where π and π' are clusters from X. This means that every set in π is also in π'. Thus, $\pi\subset\pi'$ and by (3.2), (3), $\pi=\pi'$. Hence, $\mathrm{cl}(\pi)=\pi$ for every point π in the space Y.

Now the correspondence which assigns to each point x in X the cluster π_x determined by it is a well-defined transformation mapping X into Y which we will denote by f. Note that $fA = \mathfrak{A}$ for every subset A of X, so in order to show that (4.3) holds we must show that, using (4.5),

(4.6) $A\ \delta\ B$ in X if and only if $\overline{\mathfrak{A}}$ meets $\overline{\mathfrak{B}}$ in Y.

So if $A\ \delta\ B$ there exists, by (P.6), a cluster π to which both A and B belong. Thus, by definition of $\overline{\mathfrak{A}}$, we have $\pi \in \overline{\mathfrak{A}} \cap \overline{\mathfrak{B}}$. On the other hand, if $\pi \in \overline{\mathfrak{A}} \cap \overline{\mathfrak{B}}$ then A and B are in π so that, by (C.1), $A\ \delta\ B$.

To show that $fX = \mathfrak{X}$ is regularly dense in Y suppose that α is an open subset of Y and that $\pi \in \alpha$. We thus have $\pi \in Y - \alpha = \mathrm{cl}(Y - \alpha)$. This means, by (4.4), that there exists some subset E of X such that E is in every cluster of $Y - \alpha$ but that E is not in π. Hence, by (C.3), there is a C in π such that E not δ C.

Since $\overline{\mathfrak{C}}$ is the set of all clusters to which C belongs we have $\pi \in \overline{\mathfrak{C}}$. And since E belongs to every cluster in $Y - \alpha$ and E not δ C, then C cannot belong to any cluster in $Y - \alpha$, by (C.1). Hence $\overline{\mathfrak{C}}$ is contained in α and we have shown that $\mathfrak{X}$ is regularly dense in Y.

The proof is now complete.

5. **Symmetric P_1-spaces.** A P_s-space (X, δ) in which δ satisfies the additional axiom

(5.1) $x\ \delta\ y$ implies $x = y$ for all points $x, y \in X$

is called a *symmetric P_1-space* (see [**3**]). The following theorem follows directly from (C.1) and (5.1).

(5.2) THEOREM. *Every cluster π from a symmetric P_1-space (X, δ) possesses at most one point.*

(5.3) THEOREM. *Given a set X and some binary relation δ on the power set of X, the following are equivalent*:

(I′) *There exists a T_1 topological space Y in which X can be topologically imbedded as a regularly dense subset so that* (1.1) *holds.*

(II′) *δ is a symmetric P_1-relation satisfying* (P.6).

PROOF. The demonstration is similar to that of Theorem (4.2). To see that (5.1) holds, note that $\bar{x} \cap \bar{y} \neq \emptyset$ implies that $x \cap y \neq \emptyset$, or $x = y$.

To show that our imbedding is topological we note first that, because of (5.2) the correspondence between X and $\mathfrak{X}$ induced by the identification of x with the cluster π_x determined by it is one-to-one. To see that the mapping is bicontinuous we must show that if A is a subset of X, $x \in A^{\delta}$ if and only if $\pi_x \in \mathrm{kl}(\mathfrak{A})$, where $\mathrm{kl}(\mathfrak{A})$ is the closure of $\mathfrak{A}$ in $\mathfrak{X}$ relative to the space Y.

So suppose $x \in A^{\delta}$ and that P absorbs $\mathfrak{A}$. Then P is a member of every π_a in $\mathfrak{A}$ and it follows that $a\ \delta\ P$ for every a in A. Thus, $A \subset P^{\delta}$ and since $A \in \pi_x$ we have, from (3.2), (2), that $P \in \pi_x$. Thus, $\pi_x \in \mathrm{kl}(\mathfrak{A})$.

On the other hand, suppose $\pi_x \in \mathrm{kl}(\mathfrak{A})$. Then since A absorbs $\mathfrak{A}$ we have $A \in \pi_x$, i.e., $A\ \delta\ x$ and hence $x \in A^{\delta}$. This completes the proof.

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