EXTREMAL LENGTH OF WEAK HOMOLOGY
CLASSES ON RIEMANN SURFACES

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1. The square integrable harmonic differentials on a Riemann surface \( \mathcal{W} \) form a Hilbert space \( \Gamma_h \). Let \( \Gamma_x \) be a closed subspace of \( \Gamma_h \). Let \( c \) be a 1-chain on \( \mathcal{W} \). There exists a unique element \( \psi(c) \in \Gamma_x \) with the property \( \int \omega = (\omega, \psi(c)) \) for all \( \omega \in \Gamma_x \). We refer to \( \psi(c) \) as the \( \Gamma_x \)-reproducing differential for \( c \). Accola [1] has shown that if \( c \) is a cycle, then the extremal length of the homology class of \( c \) is equal to the square of the norm of the \( \Gamma_h \)-reproducing differential for \( c \) (cf. also [3]). Two specific problems raised by Accola's result are the following. For the important subspaces \( \Gamma_x \), does the norm of the \( \Gamma_x \)-reproducing differential for a cycle have an extremal length interpretation? Secondly, we may ask for a family of curves associated with a 1-chain \( c \), not necessarily a cycle, whose extremal length gives the norm of the \( \Gamma_h \)-reproducer for \( c \). (By Abel's theorem, the vanishing of the norm of this reproducer implies that \( \partial c \) is a principal divisor.)

In the present paper we give an answer to the first question for the subspace \( \Gamma_{he} \). Theorem 1 states that an associated geometric configuration is the weak homology class of \( c \).²

2. Let \( \Gamma_x \) be a closed subspace of \( \Gamma_h \) such that \( \Gamma_x = \Gamma_z \). We say that two cycles \( c_1 \) and \( c_2 \) are \( \Gamma_x \)-homologous, denoted by \( c_1 \sim c_2 \) (mod \( \Gamma_x \)), if \( \int c_1 - c_2 \omega = 0 \) for all \( \omega \in \Gamma_x \). Denote the \( \Gamma_x \)-homology class of a cycle \( c \) by \( c^x \).

An invariant expression \( \rho(z) |dz| \) with \( \rho \) a nonnegative and lower semicontinuous function is called a linear density. The \( \rho \)-area is

\[
A(\rho) = \int \int_{\mathcal{W}} \rho^2 dx dy.
\]

The \( \rho \)-length of a family \( \mathcal{F} \) of arcs is

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² Another special case of the first question has been settled by A. Marden. He has shown that a geometric configuration for the subspace \( \Gamma_{ho} \) (notation as in [2]) is the set of relative, i.e., possibly infinite cycles which are weakly homologous to \( c \) [An extremal length problem and the bilinear relation on open Riemann surfaces, doctoral dissertation, Harvard University, May, 1962].
The extremal length of $\mathcal{F}$ is

$$\lambda(\mathcal{F}) = \sup_{\rho} L(\mathcal{F}, \rho)^2 / A(\rho).$$

**Lemma 1.** Let $c$ be a cycle on $W$. Let $\lambda(c^e)$ be the extremal length of all cycles $\Gamma_e$-homologous to $c$. Let $\psi$ be the $\Gamma_e$-reproducing differential for $c$. Then $\lambda(c^e) \geq ||\psi||^2$.

**Proof.** Let $p_0\, dz$ be the linear density $|\psi + i\psi^*|$. Then $A(p_0) = ||\psi||^2$. For $\delta \in c^e$ we have $\int p_0\, dz \geq \int \psi^* = \int |\psi| = ||\psi||^2$, and the desired inequality follows. We have used the fact that $\Gamma_e = \Gamma_z$ which implies that $\psi$ is real.

3. We shall prove a converse of Lemma 1 for $\Gamma_z = \Gamma_{heo}$. First note that $c_1 = c_2$ (mod $\Gamma_{heo}$) if and only if $c_1 - c_2$ is weakly homologous to zero. In fact, $c_1 = c_2$ (mod $\Gamma_{heo}$) holds exactly when $c_1 - c_2$ is a dividing cycle (see Theorem V.20D of [2]), which in turn is equivalent to being weakly homologous to zero (see Theorem I.32C, ibid.).

Let $\Omega$ be the interior of a compact bordered Riemann surface $\mathfrak{F}$. Let $c$ be a cycle in $\Omega$ and $\psi_0$ the $\Gamma_{heo}$-reproducer for $c$. Let $L$ be the normal operator for the canonical partition of $\partial \Omega$ (cf. [2]). Corollary 6 of [4] shows that $\psi_0 = (2\pi)^{-1} dp^*$ where $p$ is a harmonic function on $W - c$ and satisfies $p = L^* \phi$ in a boundary neighborhood of $\Omega$. Thus $p$ is constant on each contour $\beta_i$ of $\mathfrak{F}$ and $\int_{\beta_i} dp^* = 0$. If $\delta$ is a cycle on $\Omega$ then $(2\pi)^{-1} \int_{\delta} dp$ is an integer equal to the intersection number $\delta \times c$. Furthermore, the $\Gamma_{heo}$-reproducer for an open surface $W$ is the limit of $\psi_0$ for exhausting canonical subregions $\Omega \to W$.

In the course of the following proofs we find occasion to use arguments similar to those expressed or implied in Accola [1]. For convenience to the reader we repeat his reasoning in such situations.

**Lemma 2.** Let $c$ be a cycle on a compact bordered Riemann surface and $\psi$ the $\Gamma_{heo}$-reproducing differential for $c$. Then $\lambda(c_{heo}) = ||\psi||^2$.

**Proof.** Denote the bordered surface by $\mathfrak{F}$ and its interior by $\Omega$. Let the contours be $\beta_1, \ldots, \beta_n$. Let $\mathcal{U}$ be the equivalence class of $\beta_1$ in the sense of Accola [1]. That is, $\mathcal{U}$ is the set of points in $\mathfrak{F}$ which can be joined to a point of $\beta_1$ by an arc $\delta$ for which $\int_{\delta} \psi^*$ is an integer. $\mathcal{U}$ is a closed set which is locally a level curve of a harmonic function. Let $\Omega - \mathcal{U} = R_1 \cup \cdots \cup R_m$ be a decomposition into components. Each $R_i$ is a finite surface. The points of $\mathcal{U}$ serve as a piecewise analytic
We denote the Riemann surface together with its boundary by $R_r$. If we allow multiplicities for the prime ends, $R_r$ together with its boundary shall be denoted by $R^*_r$. We may omit the details of this construction but remark that there is an analytic mapping $R^*_r \to \Omega$ which restricts to the identity on $R_r$. By means of this mapping we refer to points of $\partial R^*_r$ as belonging to $\mathcal{U}$ or $\partial \Omega$.

$\psi^*$ is exact on $R_r$, say $\psi^* = dp_r$, and we know that $p_r$ extends harmonically to $R^*_r$. Since each point of $\partial R^*_r$ belongs to $\mathcal{U}$ or $\partial \Omega$ we see that $p_r$ is constant on each component of $\partial R^*_r$. We adjust $p_r$ so that the smallest such constant on $\mathcal{U}$ is zero. Now let $\rho_r$ be the collection of those boundary components of $R^*_r$ on which $p_r = 0$, $\sigma_r$ those on which $p_r = 1$, and let $\tau_r$ contain the remaining ones. We orient them so that $\partial R^*_r = \rho_r + \sigma_r + \tau_r$. The points of $\rho_r$ and $\sigma_r$ belong to $\mathcal{U}$, those of $\tau_r$ belong to $\partial \Omega \setminus \mathcal{U}$. Let us show that if $\tau_r$ contains a point $t$ of some $\beta_k$ then it must contain all of $\beta_k$. For $t \in \beta_k$, $\int_t \psi^*$ is not an integer and since $\psi^* = 0$ along $\beta_k$, it follows that $\beta_k$ has a connected neighborhood disjoint from $\mathcal{U}$. This neighborhood must be in $R_r$, hence $\beta_k \subset \tau_r$.

By means of the mapping $R^*_r \to \Omega$, we consider $\sigma_r$ as a 1-chain on $\Omega$ and claim that $c = \Sigma_r \sigma_r \mod \Gamma_{hse}$. Let $\omega \in \Gamma_{hse}$ and assume that $\omega$ extends harmonically to $\Omega$. Then $\int \omega = \langle \omega, \psi \rangle = \Sigma_r \langle \omega, dp^*_r \rangle$. By partial integration we have $\langle \omega, dp^*_r \rangle = \int d(p_r \wedge \omega) = \int p_r \omega + \int p_r dp_r \omega$. We have seen that $\tau_r$ is a union of contours $\beta_r, \ldots, \beta_n$ on each of which $p_r$ is a constant. Since $\omega$ is semiexact we obtain $\int \omega = \int \sigma_r \omega$. It follows that $c - \Sigma_r \sigma_r$ is a dividing cycle.

The function $p_r$ in $R_r$ has boundary values 0 on $\rho_r$, 1 on $\sigma_r$, and constants $k_{nu}$ on $\beta_n$, those contours of $\Omega$ which make up $\tau_r$. These constants must satisfy $0 < k_{nu} < 1$ in order for the flux condition $\int_{\partial \Omega} dp^*_r = 0$ to hold. Consequently, for $s \in (0, 1)$ the level curves $\sigma_r(s) = k^{-1}_{nu}(s)$ are compact and weakly homologous to $\sigma_r$, except for the finite number of values $s = k_{nu}$. Let $\sigma(s) = \Sigma_n \sigma_n(s)$. Let $\rho$ be a linear density on $\Omega$. Then for almost all $s \in (0, 1)$

$$L^2(\rho, c_{hse}) \leq \left( \int_{\sigma(s)} \rho \psi \right)^2 \leq \int_{\sigma(s)} \rho^2 \psi \int_{\sigma(s)} \psi = \|\psi\|^2 \int_{\sigma(s)} \rho^2 \psi.$$

Integrating over $s \in (0, 1)$ we obtain

$$L^2(\rho, c_{hse}) \leq \|\psi\|^2 A(\rho).$$

This, together with the opposite inequality of Lemma 1, completes the proof.

4. Theorem. Let $W$ be an open Riemann surface. Let $\psi$ be the $\Gamma_{hse}$-reproducing differential for a cycle $c$ on $W$. Then $\|\psi\|^2$ gives the extremal length of all cycles weakly homologous to $c$. 

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PROOF. Let $\Omega$ be a canonical subregion of $W$ and $\psi_0$ the $\Gamma_{h^*}(\Omega)$-reproducing differential for $c$. Thanks to the above lemmas we have
\[
\lambda(c^{h^*}) \geq ||\psi||^2 = \lim_{a \to \Omega} ||\psi_0||^2 = \lim_{a \to \Omega} \lambda_0(c^{h^*}) \geq \lambda(c^{h^*}).
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BIBLIOGRAPHY


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