SHORTER NOTES

The purpose of this department is to publish very short papers of an unusually elegant and polished character, for which there is normally no other outlet.

SOME SIMPLE APPLICATIONS OF THE CLOSED GRAPH THEOREM

JESÚS GIL DE LAMADRID[1]

Our discussion is based on the following two simple lemmas. Our field of scalars can be either the reals or the complex numbers.

LEMMA 1. *Let E be a Banach space under a norm $\| \|$ and F a normed space under a norm $\| \|_1$. Suppose that $\| \|_2$ is a second norm of F with respect to which F is complete and such that $\| \|_2 \geqq k\| \|_1$ for some $k>0$. Assume finally that $T: E \to F$ is a linear transformation which is bounded with respect to $\| \|$ and $\| \|_1$. Then T is bounded with respect to $\| \|$ and $\| \|_2$.*

PROOF. The product topology of $E \times F$ with respect to $\| \|$ and $\| \|_1$ is weaker (coarser) than the product topology with respect to $\| \|$ and $\| \|_2$. Since T is bounded with respect to $\| \|$ and $\| \|_1$, its graph G is $(\| \|, \| \|_1)$-closed. Hence G is also $(\| \|, \| \|_2)$-closed. Therefore T is $(\| \|, \| \|_2)$-bounded by the closed graph theorem.

LEMMA 2. *Assume that E is a Banach space under a norm $\| \|_1$ and that F is a vector subspace[2] of E, which is a Banach space under a second norm $\| \|_2 \geqq k\| \|_1$, for some $k>0$. Suppose that $T: E \to E$ is a bounded operator with respect to $\| \|_1$ and $\| \|_1$ such that $T(F) \subset F$. Then the restriction of T to F is bounded with respect to $\| \|_2$ and $\| \|_2$.*

PROOF. Since the $\| \|_2$-topology of F is stronger (finer) than its $\| \|_1$-topology, T restricted to F is $(\| \|_2, \| \|_1)$-continuous. We now apply our previous lemma with E replaced by F to get that the restriction of T to F is $(\| \|_2, \| \|_2)$-continuous.

The above two lemmas have some interesting applications to certain questions about simple Banach algebras and other questions about tensor products of Banach spaces. These applications will be discussed in a subsequent article. Here we only give some simple ap-

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[2] In the interesting cases F is not closed.

plications to more concrete situations. Typical of these is the one involved in the following theorem.

THEOREM 1. *Let T be a bounded operator on the Hilbert space $L_2([0, 1])$ so that if $\phi \in L_2([0, 1])$ is a continuous function so is $T\phi$. Then the restriction of T to $C([0, 1])$ is a bounded operator of $C([0, 1])$.*

Clearly Theorem 1 is a concrete restatement of Lemma 2 and can be rephrased in a wide variety of contexts. In fact if S is a locally compact Hausdorff space, $C(S)$ the space of all continuous scalar-valued functions on S vanishing at ∞, and μ a bounded Radon measure on S, then E in Lemma 2 can be replaced by any $L_p(\mu)$, $1 \leqq p \leqq +\infty$ and F by any $L_r(\mu)$, $p \leqq r \leqq \pm\infty$ or, in the case that μ is positive on open sets, by $C(S)$.

UNIVERSITY OF MINNESOTA