VON NEUMANN'S THEOREM ON ABELIAN
FAMILIES OF OPERATORS

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The theorem referred to in the title is the following: If \( \{A_n\} \) is a countable family of bounded commuting normal operators over an arbitrary Hilbert space (not necessarily separable), then there is a resolution of the identity \( \{E(t)\mid 0 \leq t \leq 1\} \) and a sequence of continuous functions \( \{a_n(t)\} \) such that, for all \( n \),

\[
A_n = \int_0^1 a_n(t) dE(t).
\]

The short proof below resembles but differs from von Neumann’s original proof [2; 3].

Let \( \mathfrak{A} \) be the uniformly closed algebra generated by the set \( \{I, A_n, A_n^*, n, m = 1, 2, \ldots\} \). Then, from the general theory of Banach algebras [1] we see that \( \mathfrak{A} \) and \( C(\mathcal{M}) \) are isometrically isomorphic and that the maximal ideal space \( \mathcal{M} \) of \( \mathfrak{A} \) is a compact metric space, since \( \mathfrak{A} \) is separable. Hence there is a mapping \( f: S \rightarrow \mathcal{M} \) of the Cantor set \( S \) onto \( \mathcal{M} \). For \( t \) in \([0, 1]\), let \( \tilde{E}_t(M) \) be the characteristic function of the set \( f(\{0, t\} \cap S) \subset \mathcal{M} \). Each of these sets, as the union of the compact sets \( f([0, t - 1/n] \cap S) \) is a Borel, hence since \( \mathcal{M} \) is metric, a Baire set.

In accordance with the isometric isomorphism between the set of bounded Baire functions on \( \mathcal{M} \) and a super-ring of \( \mathfrak{A} \) [1, 26 F, 26 G], \( \tilde{E}_t(M) \) corresponds to a projection \( E(t) \), and clearly \( \{E(t)\mid 0 \leq t \leq 1\} \) is a resolution of the identity.

For \( B \) in \( \mathfrak{A} \), let \( b(t) \) be defined as follows:

\[
b(t) = \begin{cases} 
\hat{B}(f(t)), & t \in S; \\
\alpha b(t_1) + \beta b(t_2), & t = \alpha t_1 + \beta t_2, \text{ where } (t_1, t_2) \text{ is one of the intervals deleted in forming } S \text{ and } 0 \leq \alpha, \beta; \alpha + \beta = 1.
\end{cases}
\]

Since \( f \) and \( \hat{B} \) are continuous, \( b(t) \) is continuous. A direct computation shows \( B = \int_0^1 b(t) dE(t) \) and, in particular, \( A_n = \int_0^1 a_n(t) dE(t) \).

Indeed, for \( \epsilon > 0 \), there is a \( \delta > 0 \) such that if \( |t_1 - t_2| < \delta \) then \( |b(t_1) - b(t_2)| < \epsilon \). Thus let \( 0 = t_0 < t_1 < \cdots < t_n = 1 \) where \( \max_i |t_{i+1} - t_i| < \delta \). For \( \tau_i \in [t_i, t_{i+1}) \) we find

\[
\text{Received by the editors December 24, 1962 and, in revised form, February 13, 1963.}
\]
For any $M \in \mathfrak{M}$, there is a unique $i_0$ such that

$$M \in f([0, i_{i_0+1}) \cap S) \setminus f([0, i_{i_0}) \cap S).$$

For this $M$, then, $\hat{E}_{i_{i_0+1}}(M) - \hat{E}_{i_0}(M) = 0$ unless $i = i_0$, in which case $\hat{E}_{i_{i_0+1}}(M) - \hat{E}_{i_0}(M) = 1$. Furthermore, $M = f(\tau)$, where $\tau \in (t_{i_0}, t_{i_0+1}) \cap S$. Thus $\hat{B}(M) = b(\tau)$ and

$$|\hat{B}(M) - b(\tau_i)| = |b(\tau) - b(\tau_i)| < \varepsilon.$$

In short, $||\hat{B}(M) - \sum_{i=0}^{n-1} b(\tau_i) [\hat{E}_{i_{i_0+1}}(M) - \hat{E}_{i_0}(M)]|| < \varepsilon$, and finally $||B - \sum_{i=0}^{n-1} b(\tau_i) [E(t_{i_{i_0+1}) - E(t_i))|| < \varepsilon$. The required conclusion then follows.

The usual extensions of the above theorem to the cases where (a) the $A_n$ are not necessarily bounded or (b) $\{A_n\}$ is replaced by a not necessarily countable family $\{A_\lambda\}$ of not necessarily bounded, commuting normal operators on a separable Hilbert space, follow readily [2].

On the other hand, let $\alpha$ be a cardinal greater than $2^{2\aleph_0}$, and let $\Lambda = \{\lambda\}$ be a set of cardinality $\alpha$. Then the set $\Lambda = \{x(\lambda) | x(\lambda) \text{ complex-valued, } \sum_{\lambda \in \Lambda} |x(\lambda)|^2 < \infty\}$ is a (highly nonseparable) Hilbert space on which the projections $P_\lambda: x(\lambda) \rightarrow y(\lambda) = x(\mu)\delta_{\lambda\mu}$, form a commuting family of bounded Hermitian operators. If there were some resolution of the identity $\{E(t)\}$ such that for $\mu \in \Lambda$, $P_\mu = \int_0^1 p_\mu(t) dE(t)$, where $p_\mu(t)$ is a complex-valued function, then the cardinality of the set $\{p_\mu(t)\}$ would be $\alpha > 2^{2\aleph_0}$, which is impossible, since the cardinality of the set of all complex-valued functions on $[0, 1]$ is $2^{2\aleph_0}$.

**Bibliography**


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