SOME RESULTS ON THE ASYMPTOTIC COMPLETION OF AN IDEAL

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1. Introduction. Let \( \mathfrak{o} \) be a commutative ring with identity. A semi-prime operation on \( \mathfrak{o} \) is a mapping \( a \rightarrow a_p \) of \( \mathfrak{o} \)-ideals into \( \mathfrak{o} \)-ideals which satisfies all the conditions of a prime operation in the sense of Krull [1; 2], except that one does not require that \( (xa)_p = x(a)_p \) for all \( x \in \mathfrak{o} \) and all ideals \( a \). Specifically, a semi-prime operation satisfies the conditions

\[
\begin{align*}
\text{(i)} & \quad a \subseteq a_p, \\
\text{(ii)} & \quad a \subseteq b \text{ implies that } a_p \subseteq b_p, \\
\text{(iii)} & \quad a_p a_p = a_p, \\
\text{(iv)} & \quad a_p b_p \subseteq (ab)_p.
\end{align*}
\]

Formal consequences [5] of the foregoing definition are (v) \( a_p = 0 \), (vi) \( (a_p b_p)_p = (ab)_p \), (vii) \( a_p \subseteq b_p \) and \( c \) arbitrary imply that \( (ac)_p \subseteq (bc)_p \), (viii) \( (\sum a_a)_p = (\sum (a_a)_a)_p \), and (ix) \( \cap a (a_a)_p = (\cap a (a_a)_p)_p \), where (viii) and (ix) are arbitrary sums and intersections, respectively.

The identity operation \( a \rightarrow a \) and the radical operation \( a \rightarrow \text{Rad} \ a \) are trivial examples of semi-prime operations. If \( \mathfrak{o} \) is an integrally closed domain and if \( a_a \) is the integral completion of \( a \) (see [1; 2]), it is well known that the \( a \)-operation \( a \rightarrow a_a \) is a prime operation. If \( \mathfrak{o} \) is an arbitrary integral domain the \( a \)-operation is still a semi-prime operation. In §6 it is shown by a variation of the classical argument that the \( a \)-operation is always a semi-prime operation, even if \( \mathfrak{o} \) is not an integral domain.

If \( a \) is an \( \mathfrak{o} \)-ideal let \( v_a(x) = n \) in case \( x \in a^n \) and \( x \notin a^{n+1} \) and let \( v_a(x) = \infty \) in case \( x \in a^m \) for all \( m \). Rees proved [4] that \( \lim_{n \to \infty} v_a(x^n)/n \) exists for all \( x \in \mathfrak{o} \) and that the function \( \bar{v}_a \) defined by \( \bar{v}_a(x) = \lim_{n \to \infty} v_a(x^n)/n \) is a homogeneous pseudo-valuation on \( \mathfrak{o} \). The asymptotic completion of the ideal \( a \) is defined to be

\[
(2) \quad a_a = \{ x \in \mathfrak{o} : \bar{v}_a(x) \geq 1 \}.
\]

In case \( \mathfrak{o} \) is noetherian, this definition of asymptotic completion of an

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ideal is equivalent to one given by Samuel [6], and furthermore, the
s-operation \( a \rightarrow a_* \) is precisely the \( a \)-operation [3; 4].

In this paper the following result is proven for the arbitrary (not
necessarily noetherian) case:

**Theorem.** The s-operation \( a \rightarrow a_* \) is always a semi-prime operation
which satisfies the cancellation law

\[(ac)_* \subseteq (bc)_* \text{ and } a \subseteq \text{Rad } c \text{ together imply that } a_* \subseteq b_*.

Moreover, \( a_* = b_* \) if and only if \( \hat{v}_a = \hat{v}_b \).

A consequence of the cancellation law (3) is that, if \( (ac)_* = (bc)_* \),
and \( a + b \subseteq \text{Rad } c \), then \( a_* = b_* \).

However, the s-operation is in general not the same as the a-
operation, although it is always true that \( a_* \subseteq a \). In particular, \( (x_0)_* \),
ned not be \( x_0 \), even if \( \sigma \) is an integrally closed domain. In §5 a char-
acterization is given of those integral domains in which all principal
ideals are asymptotically complete.²

2. **Preliminary results.** By definition of a homogeneous pseudo-
valuation, \( \hat{v}_a \) satisfies the conditions

\[(i) \quad \hat{v}_a(x \pm y) \geq \min (\hat{v}_a(x), \hat{v}_a(y)),
(ii) \quad \hat{v}_a(xy) \geq \hat{v}_a(x) + \hat{v}_a(y),
(iii) \quad \hat{v}_a(x^n) = n\hat{v}_a(x) \text{ for all positive integers } n.

It follows that \( a_* \) as defined by (2) is an ideal. Also, one easily sees
that

\[(i) \quad a \subseteq b \text{ implies that } \hat{v}_a \leq \hat{v}_b,
(ii) \quad \hat{v}_a \leq \hat{v}_b \text{ implies that } a_* \subseteq b_*.

**Lemma 1.** A necessary and sufficient condition that \( \hat{v}_a(x) \geq \alpha > 0 \) is
that for every rational number \( 0 < p/q < \alpha \) there exists a positive integer
\( k \) such that \( x^{pk} \subseteq a^{n_k} \). In particular, if \( x \in a_*, \) then for every positive
integer \( n \), there exists a positive integer \( k \) such that \( x^{(n+1)k} \subseteq a^{nk} \).

**Proof.** \( \hat{v}_a(x) = \sup \{ \hat{v}_a(x^n)/n : n \text{ a positive integer} \} \).

**Proposition 2.** If \( a \) and \( b \) are ideals and if \( x \in a \), then

\[(i) \quad \frac{1}{\hat{v}_{ab}(x)} \leq \frac{1}{\hat{v}_a(x)} + \frac{1}{\hat{v}_b(x)},
(ii) \quad \hat{v}_a^n(x) = \frac{1}{n} \hat{v}_a(x), \quad n \text{ a positive integer}.

² An ideal \( a \) is asymptotically complete, or s-complete, in case \( a_* = a \).
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Proof. Assume $x \in \text{Rad } ab$, for otherwise (i) reduces to $\infty \leq \infty$. This implies that $\tilde{v}_a(x) = \alpha > 0$ and $\tilde{v}_b(x) = \beta > 0$. If $0 < n/m < \alpha$ and $0 < p/q < \beta$ then by Lemma 1 there exist $j$ and $k$ such that $x^{n+1} \in a^{n+1}$ and $x^{k+1} \in b^{k+1}$. It follows that $x^{(n+1)p+q} \in (ab)^{n+p+q}$. Hence $\tilde{v}_{ab}(x) \geq np/(mp+nq)$. To obtain (i) let $n/m \to \alpha$ and $p/q \to \beta$.

If $0 < p/q < \tilde{v}_a(x)$ then for some $k$, $x^{k+1} \in \tilde{a}^{nk}$. This implies that $\tilde{v}_a(x) \geq n(p/q)$, which in turn implies that $\tilde{v}_a(x) \geq n\tilde{v}_a(x)$. The opposite inequality required for (ii) follows from (i).

Proposition 3. If $a$ and $b$ are ideals, then for all $x \in 0$, $\lim_{n \to \infty} n\tilde{v}_a\tilde{v}_b(x)$ exists. Moreover, the value of the limit is $\tilde{v}_a(x)$ if $x \in \text{Rad } b$, and is zero otherwise. In particular, if $a \subseteq \text{Rad } b$, then for all $x \in 0$,

$$\lim_{n \to \infty} n\tilde{v}_a\tilde{v}_b(x) = \tilde{v}_a(x).$$

Proof. Since $a^{n}b \subseteq a^{n}$ it follows by (5) and Proposition 2 that $n\tilde{v}_a\tilde{v}_b \leq n\tilde{v}_a = \tilde{v}_a$. On the other hand, if $x \in \text{Rad } b$, it is seen from Proposition 2 that

$$\lim_{n \to \infty} \frac{1}{n\tilde{v}_a\tilde{v}_b(x)} \leq \lim_{n \to \infty} \left[ \frac{1}{\tilde{v}_a(x)} + \frac{1}{n\tilde{v}_b(x)} \right] = \frac{1}{\tilde{v}_a(x)}.$$

Thus $\tilde{v}_a(x) \leq \lim \inf_{n \to \infty} n\tilde{v}_a\tilde{v}_b(x) \leq \lim \sup_{n \to \infty} n\tilde{v}_a\tilde{v}_b(x) \leq \tilde{v}_a(x)$.

3. Proof of Theorem. From definition (2) and conditions (5) it follows that the $s$-operation satisfies (i) and (ii) of (1). It now will be shown that (iv) of (1) holds, that is

$$(a)\ ^n \subseteq (ab)^n.$$

For, suppose that $x \in a$ and $y \in b$. If $n$ is a positive integer there exists by Lemma 1 a positive integer $k$ such that $x^{(n+1)k} \subseteq a^{nk}$ and $y^{(n+1)k} \subseteq b^{nk}$. Hence $(xy)^{(n+1)k} \subseteq (ab)^{nk}$. This implies that $\tilde{v}_{ab}(xy) \geq 1$, which establishes (6). A consequence of (6) is that by induction, for every positive integer $n$,

$$(a^n)_n \subseteq (a^n)_s.$$

To establish that the $s$-operation satisfies condition (iii) of (1) it will be sufficient to show that for all $x \in 0$

$$\tilde{v}_a(x) = \tilde{v}_a(x).$$

Suppose $0 < r/t < \tilde{v}_a(x)$. It follows that for suitable $k$,

$$x^{rk} \in (a)^{rk} \subseteq (a^{rk})_s.$$

Hence $\tilde{v}_a(x) = rk\tilde{v}_a(x) = (r/t)\tilde{v}_a(x^{rk}) \geq r/t$. It is thus seen that $\tilde{v}_a(x)$
The opposite inequality required for (7) follows from the fact that \( a \subseteq a_s \).

Now it will be shown that the cancellation law (3) holds. From \((ac)_s \subseteq (bc)_s\), it follows that

\[(a^c)_s \subseteq (abc)_s \subseteq (b^c)_s,\]

and by induction it is seen that

\[(a^c)_s \subseteq (b^n c)_s.\]

Hence for all \( n \), \((a^c)_s \subseteq (b^n c)_s\), which implies that

\[\bar{v}_{a^n c} \leq \bar{v}_{b^n} = \frac{1}{n}.\]

Since \( a \subseteq \text{Rad } c \), Proposition 3 yields that

\[\bar{v}_a = \lim_{n \to \infty} n\bar{v}_{a^n c} \leq \bar{v}_b.\]

Hence \( a_s \subseteq b_s \).

The last statement of the Theorem follows from (7).

4. Remarks. A principal ideal need not be \( s \)-complete, even if \( o \) is an integrally closed domain. For example, let \( K[x, y] \) be the polynomial ring in two indeterminates over a field \( K \). Consider the valuation \( v \) defined on \( K[x, y] \) as follows: If \( f(x, y) = \sum a_{ij}x^iy^j \) then \( v(f) = \min \{ (i, j) : a_{ij} \neq 0 \} \) where the pairs \( (i, j) \) are ordered lexicographically. The associated valuation ring \( R_v \) is a rank 2, discrete valuation ring. The maximal ideal of \( R_v \) is \( M_v = yR_v \). The other nonzero prime ideal is \( P = (xy^{-m} : m = 1, 2, \ldots )_{R_v} \). Obviously \( xR_v \) is a proper subideal of \( P \). However, \((xR_v)_s = P \). To show this it will be sufficient to show that \( xy^{-m} \in (xR_v)_s \) for all positive \( m \). Clearly for any \( n \geq 1 \), \((xy^{-m})^{n+1} = x^n(xy^{-m(n+1)})\), and hence \((xy^{-m})^{n+1} \in x^nR_v \). This implies that \( xy^{-m} \in (xR_v)_s \).

The radical restriction in the cancellation law (3) is essential. Consider the valuation ring \( R_v \) in the foregoing example. It is easy to verify that \( M_vP = P = M_v^2P \), and that \((M_v^n)_{s} = M_v^n \) for all \( n \). Hence \((M_vP)_s \subseteq (M_v^2P)_s \), but \((M_v)_s \not\subseteq (M_v^2)_s \). However, it is noted that \( M_v \subseteq \text{Rad } P = P \).

5. A result for integral domains. In §4 it was pointed out that in general a principal ideal was not \( s \)-complete. The following proposition characterizes those integral domains in which all principal ideals are \( s \)-complete. Let \( K \) be the field of quotients of an integral domain \( o \). An element \( a \in K \) is almost integral over \( o \) in case there exists a non-
zero \( y \in \mathfrak{o} \) such that \( y \alpha^n \in \mathfrak{o} \) for all \( n \). The set of all elements in \( K \) which are almost integral over \( \mathfrak{o} \) forms an overring \( \hat{\mathfrak{o}} \) of \( \mathfrak{o} \). If \( \hat{\mathfrak{o}} = \mathfrak{o} \) then \( \mathfrak{o} \) is said to be \textit{completely integrally closed}. See \[1\].

**Proposition 4.** All principal ideals in an integral domain \( \mathfrak{o} \) are \( s \)-complete if and only if \( \mathfrak{o} \) is completely integrally closed.

**Proof.** Assume \( \mathfrak{o} \) is completely integrally closed. If \( x, y \neq 0 \) and \( y \in (x_0)_n \), then for each positive integer \( n \) there exists a positive integer \( k_n \) such that \( y^{(n+1)k_n} \in x^{n+1} \mathfrak{o} \). It follows that \( (y^{n+1}/x^n)k_n \in \mathfrak{o} \), and hence \( y^{n+1}/x^n \) is integral over \( \mathfrak{o} \) for each \( n \). But \( \mathfrak{o} \) completely integrally closed implies that \( \mathfrak{o} \) is integrally closed. Thus for all \( n \), \( y(y/x)^n = y^{n+1}/x^n \in \mathfrak{o} \). From the hypothesis it follows that \( y/x \in \mathfrak{o} \), and hence \( y \in \mathfrak{o} \).

On the other hand assume that every principal ideal is \( s \)-closed. If \( y/x \in K \) is almost integral over \( \mathfrak{o} \), then for some nonzero \( z \in \mathfrak{o} \), \( z(y/x)^n \in \mathfrak{o} \) for all \( n \). Hence \( zy^n \in x^n \mathfrak{o} \) for all \( n \). It follows that \( (zy)^n + 1 = yz^n(y/x)^n \in y^n x^n \mathfrak{o} \subseteq (xz)^n \mathfrak{o} \). This implies that \( zy \in (x_0)_n = x_0 \mathfrak{o} \). Hence \( y/x \in \mathfrak{o} \).

6. **The \( a \)-operation.** An element \( x \in \mathfrak{o} \) is \textit{integral over} an ideal \( a \) in case for some \( n \) there exist \( a_i \in \mathfrak{a}^i, i = 1, \ldots, n \), such that

\[
x^n + a_1 x^{n-1} + \cdots + a_n = 0.
\]

The set of all elements integral over \( a \) is denoted by \( a_{\alpha} \). In case \( \mathfrak{o} \) is an integral domain it is well known \[2\] that

\[ x \in a_{\alpha} \text{ iff there exists a finitely generated nonzero ideal } b \text{ such that } xb \subseteq ab. \]

From this it easily follows that \( a_{\alpha} \) is an ideal, called the integral completion of \( a \), and that the \( a \)-operation \( a \rightarrow a_{\alpha} \) is a semi-prime operation for which the following cancellation law holds:

\textit{If both } (ac)_{\alpha} \subseteq (bc)_{\alpha} \text{ and } c_{\alpha} = c_{\alpha'} \text{ for some finitely generated nonzero ideal } c', \text{ then } a_{\alpha} \subseteq b_{\alpha}. \]

In case \( \mathfrak{o} \) is not necessarily an integral domain a variation of the classical argument shows the following modification of (8) to hold:

\[ x \in a_{\alpha} \text{ iff there exists a finitely generated ideal } b \text{ such that both } x \in \text{Rad } b \text{ and } xb \subseteq ab. \]

\[ (i) \text{ (ii) The ideal } b \text{ above can be chosen so that its radical contains any finite number of elements in the radical of } a. \]

From (9) it still follows that \( a_{\alpha} \) is an ideal and that the \( a \)-operation is a
semi-prime operation. Moreover, the following cancellation law, shown by Nagata [3] to hold in the noetherian case, holds in general:

If \((ac)_{a} \subseteq (bc)_{a}\), if \(c_{a} = c'_{a}\) for some finitely generated ideal \(c'\), and if \(a\) is contained in every minimal prime ideal of \((0)\) in which \(c\) is contained, then \(a_{a} \subseteq b_{a}\).

Finally, it will be noted that from (9(i)) and (3) it follows that \(a_{a} \subseteq a_{a}\) always. This result also follows directly from the definition of \(a_{a}\). See [4; 5].

References


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