THE SCHNIRELMANN DENSITY OF THE SQUAREFREE INTEGERS

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It is a familiar and elementary process to show that every natural number greater than one is the sum of two squarefree natural numbers: one shows that $A(x)/x$ exceeds 1/2 for all $x \geqq 1$, where $A(x)$ is the number of squarefree natural numbers not greater than x. This crude estimate follows from the fact that $A(n) > n(1 - \sum p^{-2})$. It is also elementary that $A(x)/x \to 6/\pi^2$ as $x \to \infty$, and early numerical evidence might lead one to believe that $6/\pi^2$ is also the Schnirelmann density of this sequence, the infimum of $A(x)/x$ in the range $1 \leqq x \leqq \infty$. The purpose of this note is to prove that this is not the case.

THEOREM. *Let* $A(x) = \sum_{1 \leq n \leq x} |\mu(n)|$. *Then, for all* $x \geqq 1$, *we have*

$$A(x)/x \geqq A(176)/176 = 53/88,$$

with equality required only for $x = 176$.

PROOF. Since $53/88 < 0.603 < 0.607 < 6/\pi^2$, the proof will amount to finding a useful estimate of the smallness of $A(x)/x - 6/\pi^2$, so as to reduce the problem of finding the minimum value of $A(x)/x$ to a finite range. The computer does the rest. We begin with the usual sieve process:

$$A(x) = \sum_{n \leq x;\, d^2 \nmid n;\, 1 < d^2 \leq x} 1 = \sum_{d^2 \leq x} \mu(d) \left(\sum_{n \leq x;\, d^2 | n} 1 \right)$$
$$= \sum_{d^2 \leq x} \mu(d) \left[\frac{x}{d^2} \right].$$

Hence,

$$A(x)/x = \sum_{d^2 \leq x} \frac{\mu(d)}{d^2} + \theta \cdot \frac{A(\sqrt{x})}{x}$$
$$= \frac{1}{\zeta(2)} - \sum_{d^2 > x} \frac{\mu(d)}{d^2} + \theta \cdot \frac{A(\sqrt{x})}{x},$$

for some θ in the range $|\theta| \leqq 1$. It is known that $A(x)/x = 6/\pi^2 + o(x^{-1/2})$ (as in Landau, *Primzahlen*, p. 606), but we need exact numerical estimates. For this it seems best to proceed thus:

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$$\left| A(x)/x - 6/\pi^2 \right| \leqq \sum_{d^2>x} \frac{|\mu(d)|}{d^2} + \frac{A(\sqrt{x})}{x},$$

and we have

$$\sum_{d^2>x} \frac{|\mu(d)|}{d^2} = \int_{\sqrt{x}}^{\infty} \frac{dA(u)}{u^2} = -\frac{A(\sqrt{x})}{x} + 2\int_{\sqrt{x}}^{\infty} \frac{A(u)}{u^3}\, du.$$

Hence,

$$\left| A(x)/x - 6/\pi^2 \right| \leqq 2\int_{\sqrt{x}}^{\infty} \frac{A(u)}{u^3}\, du. \tag{1}$$

Putting the crude estimate $A(u) \leqq u$ in the right side of this inequality gives $A(x)/x - 6/\pi^2 \leqq 2/\sqrt{x}$. Now put this again into the right side of (1):

$$\begin{aligned} \left| A(x)/x - 6/\pi^2 \right| &\leqq 2\int_{\sqrt{x}}^{\infty} \left(\frac{6}{\pi^2 u^2} + \frac{2}{u^{5/2}} \right) du \\ &\leqq \frac{12}{\pi^2 \sqrt{x}} + \frac{8}{3x^{3/4}} \cdot \end{aligned} \tag{2}$$

We could continue feeding this back into (1), but this is not necessary for our purpose. From (2) we know that $\lim_{x\to\infty} A(x)/x = 6/\pi^2$, but numerical investigation shows that $A(176) = \sum_{d \leqq 13} \mu(d) [176/d^2] = 106$, and we have already remarked that $106/176 < 6/\pi^2$. To find the inf of $A(x)/x$, we need only check until x becomes so large that $\left| A(x)/x - 6/\pi^2 \right| < 6/\pi^2 - 53/88$. Now, we have $6/\pi^2 - 53/88 > .607921 - .602273$, and so we can discard all x for which $\left| A(x)/x - 6/\pi^2 \right| < .005648$. Hand computation shows that the right side of inequality (2) is less than .00555 for all $x \geqq 250^2$, so the computer was asked to check $A(x)/x$ up to this point. It was found that the minimum value was taken only at $x = 176$, when we have

$$A(x)/x = 53/88 = .60227273 \cdots .$$

In concluding, I wish to thank Professor Ernst Straus for helpful conversations about this work. Thanks are due to Mr. Alex Hurwitz for putting the numerical work on the computer at UCLA.

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