2. ———, *On the Lerch zeta function*, Pacific J. Math 1 (1951), 161–167.
3. ———, *Mathematical analysis*, Addison-Wesley, Reading, Mass., 1957.
4. Shô Iseki, *The transformation formula for the Dedekind modular function and related functional equations*, Duke Math. J. 24 (1957), 653–662.
5. ———, *A proof of a functional equation related to the theory of partitions*, Proc. Amer. Math. Soc. 12 (1961), 502–505.
6. H. Rademacher, *Zur Theorie der Modulfunktionen*, J. Reine Angew. Math. 167 (1932), 312–336.

California Institute of Technology

ON THE HOMOTOPY CLASSES OF SELF-MAPPINGS OF BORDERED RIEMANN SURFACES

CLIFFORD J. EARLE[1]

1. Let W be a Riemann surface which can be represented as the interior of a bordered surface W_0 with border B. Let F: $W \rightarrow W$ be a mapping which has continuous boundary values in B. Then F can be extended by symmetry to a self-mapping F_* of the doubled surface W_*. Let the given mapping F be homotopic to the identity map of W onto itself. It is natural to ask whether F_* is homotopic to the identity also. In this note we give an affirmative answer for a large number of cases.

2. To be specific we assume that the universal covering surface of W is the disk $U = \{z: |z| < 1\}$. W is of the form $W = U/G$ where G is a Fuchsian group of linear transformations on U. G is a group of the second kind; that is, the region of discontinuity Ω of G contains points on the circumference ∂U of U. W is thus the interior of the bordered surface W_0 whose border B corresponds to $\Omega \cap \partial U$. The double of W_0 is the surface $W_* = \Omega/G$. We assume that G has at least three limit points, so that the universal covering surface of W_* and Ω is the disk.

It is well known [1, pp. 98–99] that every mapping F: $W \rightarrow W$ is induced by a mapping f: $U \rightarrow U$. That is, $F\phi = \phi f$ where ϕ: $U \rightarrow W$ is the natural projection. Moreover, F is homotopic to the identity mapping of W onto W if and only if f can be so chosen that $Af = fA$ for all A in G.

Received by the editors March 25, 1963.
[1] This research was supported in part by the Air Force Office of Scientific Research.

THEOREM 1. *Let* $F: W \to W$ *be homotopic to the identity and let* $f: U \to U$ *induce* F *and satisfy* $Af = fA$ *for all* A *in* G. *The following are equivalent*:

1. F *has a continuous symmetric extension* F_* *to* W_* *which is homotopic to the identity under a homotopy which maps* B *into itself at each stage.*
2. f *is continuous on* $\Omega \cap \partial U$ *and maps each arc of* $\Omega \cap \partial U$ *into itself.*

PROOF. It is clear that 1 implies 2. Indeed, restricting each mapping in the given homotopy to W we obtain a family of mappings $F_t: W \to W_*$ such that F_0 is the identity and $F_1 = F$. F_t is induced by $f_t: U \to \Omega$. The f_t can be chosen so that f_t is continuous in z and t, and $Af_t = f_tA$ for all A in G [**1**, pp. 99–100]. With this choice of f_t, f_0 is the identity and $f_1 = f$. Since F_t is continuous on B and maps B into itself, f_t is continuous on $\Omega \cap \partial U$ and maps each arc α of $\Omega \cap \partial U$ into $\Omega \cap \partial U$. Since f_0 is the identity, $f_0(\alpha) = \alpha$. Therefore $f_t(\alpha) \subset \alpha$ for all t, in particular for $t = 1$ as required.

Conversely let f satisfying condition 2 be given. We extend f to all of Ω by defining $f(1/\bar{z}) = 1/\bar{f}(z)$ for z in U. By condition 2, f is continuous on Ω. Moreover f induces the self-mapping F_* of W_*. We shall show that F_* is homotopic to the identity by defining mappings $f_t: \Omega \to \Omega$ which induce the homotopy.

Following a suggestion of Professor Ahlfors we introduce the hyperbolic metric on Ω. (This is possible because the universal covering surface of Ω is U.) Since this metric has negative curvature, for any z in Ω there is a unique geodesic joining z to $f(z)$. For z in U (ext U) this geodesic lies wholly in U (ext U), and for z in ∂U the required geodesic is the arc joining z to $f(z)$ in $\Omega \cap \partial U$. We define $f_t(z)$ to be that point on the geodesic from z to $f(z)$ whose distances from z and $f(z)$ stand in the proportion $t: 1-t$. Obviously f_t is continuous in z and t, f_0 is the identity, and $f_1 = f$. Moreover, f_t maps each arc of $\Omega \cap \partial U$ into itself.

We must show that f_t induces a mapping $F_t: W_* \to W_*$. It suffices to show that $f_tA = Af_t$ for all A in G. But $f_t(A(z))$ is a point on the geodesic from $A(z)$ to $f(A(z)) = A(f(z))$, and this geodesic is the image under the isometry A of the geodesic from z to $f(z)$. Q.E.D.

3. From Theorem 1 we can obtain several conditions which imply that the extended mapping F_* is homotopic to the identity. For instance, suppose f is continuous at each point of ∂U. Then it follows from the equation $Af = fA$ for A in G that f leaves each limit point of G fixed. In particular, if α is an arc of $\Omega \cap \partial U$, each endpoint of α is left fixed, so that f satisfies condition 2.

Among the mappings $f\colon U \to U$ which have continuous extensions to ∂U are the quasiconformal mappings of U onto itself. This proves

THEOREM 2. *Let $F\colon W \to W$ be a quasiconformal homeomorphism which is homotopic to the identity. F has a continuous symmetric extension F_* to the double of W. F_* is homotopic to the identity under a homotopy leaving each border contour of W fixed.*

For another example let the border contours of W_0 be compact and let F map each contour of B into itself. It follows that f is continuous on $\partial U \cap \Omega$ and that f maps an arc α of $\partial U \cap \Omega$ into some equivalent arc $T(\alpha)$, T in G. We claim that $T(\alpha) = \alpha$. It suffices to find one common endpoint. Since α lies over a compact contour of B, there is a hyperbolic transformation A in G with $A(\alpha) = \alpha$. Consider a fixed z in α, $f(z)$ in $T(\alpha)$. As $n \to \infty$, $A^n f(z)$ approaches a fixpoint ζ of A. But $A^n f(z) = f(A^n(z)) \in T(\alpha)$, so ζ is an endpoint of $T(\alpha)$. Since $A^n z \to \zeta$, ζ is an endpoint of α as claimed.

THEOREM 3. *Let $W_0 = W \cup B$ be a bordered Riemann surface with interior W and border B consisting of compact contours. Let $F\colon W \to W$ have a continuous extension mapping each contour of B into itself. Let F_* be the symmetric extension of F to the double. Let F be homotopic to the identity. Then F_* is homotopic to the identity under a homotopy leaving each contour of B fixed.*

REFERENCE

1. L. Bers, *Analytic functions*, Princeton Univ. Press, Princeton, N. J., 1960; pp. 89–119.

HARVARD UNIVERSITY