A CONVERGENCE CRITERION FOR FOURIER SERIES

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1.1. It is known that the condition

$$(1) \qquad \omega(x_0, h) \equiv f(x_0 + h) - f(x_0) = o\left(\frac{1}{|\log h|}\right), \qquad h \to 0,$$

for a special x_0 does not imply convergence of the Fourier series $S[f]$ of $f(x)$ for that x_0. The Hardy-Littlewood convergence test [**4**, p. 63] gives the convergence of $S[f]$ at x_0, provided that the condition (1) is satisfied, and the coefficients of $S[f]$, a_n and b_n are $O(n^{-\theta})$ for some positive θ.

The aim of the note is to give a sufficient condition on the function $\omega(x_0, h)$ which will insure the convergence of the $S[f]$ at the point x_0. A similar condition was given by the author [**3**] under some additional assumptions. The first object of this note is to give a direct proof for this result.

Following A. Zygmund [**4**, p. 186] we shall use the following definition: A positive function $L(t)$ defined for $0<t<\epsilon$ will be called *slowly varying* at the point $t=0$, if, for any $\delta>0$, $L(t)t^{\delta}$ is ultimately a decreasing and $L(t)t^{-\delta}$ increasing function of t for $t \to 0$. From this definition we deduce immediately: If $L(t)$ is slowly varying, then

$$(2) \qquad \frac{L(\lambda t)}{L(\lambda)} \to 1, \qquad \text{as } \lambda \to 0,$$

for every fixed $t>0$, and even uniformly in every interval $a \leqq t \leqq b$, $a>0$, $b<\infty$.

Let $f(x)$ be a periodic function with period 2π and L-integrable over $[-\pi, \pi]$, $s_n(x)$ nth partial sum of its Fourier series. In §2.1 we shall prove the following theorem.

THEOREM I. *If at the point* x_0

$$f(x_0 + t) + f(x_0 - t) - 2f(x_0) = \phi(x_0, t) = \phi(t) \to 0, \qquad t \to 0,$$

and for $t \to 0$, $\phi(t) \equiv L(t)$ *is slowly varying then* $s_n(x_0) \to f(x_0)$, $n \to \infty$.

1.2. Let $f(x)$ be a continuous function of period 2π. It is known that $s_n(x) - f(x) = O(n^{-\alpha} \log n)$ uniformly in x if $f(x)$ belongs to Lip α, i.e., if

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$$\sup_{0<|h|\leqq\delta} |f(x+h)-f(x)| = O(\delta^\alpha), \qquad 0<\alpha\leqq 1.$$

The factor $\log n$ cannot be omitted even under the additional condition that f is of bounded variation [2]. Following Salem and Zygmund [2; 4, p. 64] the function $f(x)$ will be called of monotonic type if $f(x)+Cx$ is monotonic for suitable constant C. Then, if f is of monotonic type and belongs to Lip α, $0<\alpha<1$, $s_n(x)-f(x)=O(n^{-\alpha})$ uniformly in x. In §2.2 we shall prove the following extension of the theorem of Salem and Zygmund.

THEOREM II. *Let $f(x)$ be a continuous function over $[-\pi, \pi]$ and of monotonic type, with*

$$\omega(x,t) = \omega(t) = \sup_{0<|h|\leqq t} |f(x+h)-f(x)| = L(t), \quad t\to 0, x\in[-\pi,\pi];$$

then

$$|s_n(x)-f(x)| = O\left(L\left(\frac{\pi}{n}\right)\right) = O\left(\omega\left(\frac{\pi}{n}\right)\right).$$

2.1. PROOF OF THEOREM I. Without loss of generality, we may suppose that $x_0=0$, f is even, and $f(0)=0$. Using the known formula for partial sum $s_n(x)$ of the Fourier series $S[f]$ [4, p. 55] we have

$$s_n(0) = \frac{2}{\pi}\int_0^\epsilon f(t)\frac{\sin nt}{t}\,dt + o(1). \tag{3}$$

From the fact that $f(t)$ is a slowly varying function in the neighbourhood of $t=0$, (3) can be written as

$$\frac{2}{\pi}\int_0^\epsilon f(t)\frac{\sin nt}{t}\,dt = \frac{2}{\pi}\int_0^{\phi(n)} L(t)\frac{\sin nt}{t}\,dt + \frac{2}{\pi}\int_{\phi(n)}^\epsilon f(t)\frac{\sin nt}{t}\,dt$$
$$= \mathfrak{J}_1 + \mathfrak{J}_2,$$

where $\phi(n)=\sqrt{(\omega_1(\pi/n))}\to 0$, $n\to\infty$ and $\omega_1(\delta)=\omega_1(\delta, f)$ denotes the integral modulus of continuity of f, i.e.,

$$\omega_1(\delta, f) = \sup_{0<|h|\leqq\delta}\int_{-\pi}^{\pi} |f(t+h)-f(t)|\,dt.$$

Take $nt=\tau$, $n^{-1}=\lambda$, $n\phi(n)=\Lambda$ and we have for $\mathfrak{J}_1$

$$(\pi/2)\mathfrak{J}_1 = \int_0^\Lambda L(\lambda t)\frac{\sin t}{t}\,dt.$$

We split $\mathfrak{J}_1$ into three parts[1]

$$
\begin{aligned}
(\pi/2)\mathfrak{J}_1 &= \int_0^\delta L(\lambda t)\frac{\sin t}{t}\,dt + \int_\Delta^\Lambda L(\lambda t)\,\frac{\sin t}{t}\,dt \\
&\quad + L(\lambda)\int_\delta^\Delta\left[\frac{L(\lambda t)}{L(\lambda)} - 1\right]\frac{\sin t}{t}\,dt + L(\lambda)\int_\delta^\Delta \frac{\sin t}{t}\,dt \qquad (4)\\
&= I_1 + I_2 + I_3 + KL(\lambda),
\end{aligned}
$$

with arbitrary but fixed δ and Δ and $K=K(\delta,\Delta)=\int_\delta^\Delta t^{-1}\sin t\,dt$ for every δ and Δ. Assuming that $0<\eta<1$, we have

$$I_1 = \lambda^{-\eta}\int_0^\delta (\lambda t)^\eta L(\lambda t)\,\frac{\sin t}{t^{1+\eta}}\,dt,$$

and since $L(t)$ is slowly varying, the last integral can be majorized by

$$|I_1| \leqq \delta^\eta L(\lambda\delta)\int_0^\delta \frac{|\sin t|}{t^{1+\eta}}\,dt, \qquad 0<\eta<1, \tag{5}$$

and so $I_1\to 0$ for $\lambda\to 0$.

Similarly with $0<\eta<1$ we obtain

$$I_2 = \int_\Delta^\Lambda L(\lambda t)\,\frac{\sin t}{t}\,dt = \lambda^\eta\int_\Delta^\Lambda (\lambda t)^{-\eta}L(\lambda t)\,\frac{\sin t}{t^{1-\eta}}\,dt.$$

Using the fact that $(\lambda t)^{-\eta}L(\lambda t)$ is a decreasing function in $\Delta<t<\Lambda$, and applying the second mean value theorem, we get

$$I_2 = \frac{L(\lambda\Delta)}{\Delta^\eta}\int_\Delta^{\Lambda_1}\frac{\sin t}{t^{1-\eta}}\,dt, \qquad \Delta \leqq \Lambda_1 \leqq \Lambda, \tag{6}$$

and the last integral being convergent, we have $I_2\to 0$, $\lambda\to 0$.

Finally, since δ and Δ are fixed, the passage to the limit $\lambda\to 0$ under the integral sign in

$$I_3 = L(\lambda)\int_\delta^\Delta\left[\frac{L(\lambda t)}{L(\lambda)} - 1\right]\frac{\sin t}{t}\,dt, \tag{7}$$

is justified by (2), and we have $I_3=o(1)$, and consequently by (5), (6) and (7) $\mathfrak{J}_1=o(1)$, $n\to\infty$.

Now, consider the integral

$$(\pi/2)\mathfrak{J}_2 = \int_{\phi(n)}^{\epsilon} f(t)\,\frac{\sin t}{t}\,dt.$$

[1] For a similar decomposition see [1].

Putting $t=\tau+\pi/n$, we have

$$\pi\cdot\mathfrak{J}_2 = -\int_{\phi(n)-\pi/n}^{\phi(n)} \frac{f(t+\pi/n)}{t+\pi/n}\sin nt\,dt$$
$$+\int_{\phi(n)}^{\epsilon}\left(\frac{f(t)}{t}-\frac{f(t+\pi/n)}{t+\pi/n}\right)\sin nt\,dt+\int_{\epsilon-\pi/n}^{\epsilon}\frac{f(t+\pi/n)}{t+\pi/n}\sin nt\,dt$$
$$= -K_1+K_2+K_3$$

say. In the neighbourhood of $t=0$, $f(t)$ tends to 0 as $t\to 0$, and K_1 can be majorized by

$$(8)\qquad |K_1| \leqq \frac{|f(\phi(n)+\pi/n)|}{\phi(n)}\cdot\frac{\pi}{n} = o\left(\frac{1}{n\phi(n)}\right),\quad n\to\infty.$$

Since ϵ is fixed and the integrand in K_3 ultimately bounded, we have $K_3=o(1)$, $n\to\infty$. Now, K_2 may also be written as

$$K_2 = \int_{\phi(n)}^{\epsilon}\frac{f(t)-f(t+\pi/n)}{t}\sin nt\,dt$$
$$+\int_{\phi(n)}^{\epsilon}\left\{\frac{1}{t}-\frac{1}{t+\pi/n}\right\}f\left(t+\frac{\pi}{n}\right)\sin nt\,dt$$
$$= K_{21}+K_{22}.$$

Applying in the last integral the second mean value theorem to $\{\cdot\}$, we find

$$(9)\qquad |K_{22}| \leqq \frac{\pi}{n\phi^2(n)}\left|\int_{\phi(n)}^{\epsilon_1} f\left(t+\frac{\pi}{n}\right)\sin nt\,dt\right| = o\left(\frac{1}{n\phi^2(n)}\right).$$

Finally,

$$(10)\qquad \begin{aligned} |K_{21}| &\leqq \int_{\phi(n)}^{\epsilon}\frac{|f(t)-f(t+\pi/n)|}{t}\,dt \\ &\leqq \frac{1}{\phi(n)}\int_{\phi(n)}^{\epsilon}\left|f(t)-f\left(t+\frac{\pi}{n}\right)\right|dt \leqq \frac{1}{\phi(n)}\,\omega_1\left(\frac{\pi}{n}\right).\end{aligned}$$

If we choose $\phi(n)=\surd(\omega_1(\pi/n))$, it follows from (8), (9) and (10) that $\mathfrak{J}_2=o(1)$, $n\to\infty$, and this proves Theorem I.

2.2. PROOF OF THEOREM II. With the same reasoning as in [4, p. 64] we can suppose that $g(x)=f(x)+Cx$ is increasing. The difference $s_n(x)-f(x)$ is given by

$$\pi^{-1}\int_{-\pi}^{\pi}(f(x+t)-f(x))D_n(t)dt,$$

where $D_n(t)$ denotes the Dirichlet kernel. Replacing $f(x)$ by $g(x)$ [**4**, p. 64] we obtain for the last integral

$$\pi^{-1}\int_{-\pi}^{\pi}(g(x+t)-g(x))D_n(t)dt,$$

and we can consider only the integral over $(0, \pi)$. The proof for the remaining part is the same.

Let

$$\mathfrak{J} = \pi^{-1}\int_0^{\pi}(g(x+t)-g(x))D_n(t)dt = \pi^{-1}\left(\int_0^{\phi(n)} + \int_{\phi(n)}^{\epsilon} + \int_{\epsilon}^{\pi}\right).$$

In the last integral over (ϵ, π) the function

$$\{g(x+t)-g(x)\}(2\sin(t/2))^{-1}$$

is of bounded variation with respect to t, and this integral is therefore $O(1/n)$. We have

$$\text{(11)}\qquad \pi\mathfrak{J} = \int_0^{\phi(n)} + \int_{\phi(n)}^{\epsilon} + O\left(\frac{1}{n}\right) = \mathfrak{J}_1 + \mathfrak{J}_2 + O\left(\frac{1}{n}\right).$$

For ϵ sufficiently small but fixed, $\Phi(t) = \{(\sin t)^{-1} - t^{-1}\}$ is increasing over $\phi(n) < t < \epsilon$ and by the second mean value theorem we have

$$\int_{\phi(n)}^{\epsilon}\{g(x+t)-g(x)\}\left\{\frac{1}{2\sin(t/2)} - \frac{1}{t}\right\}\sin\left(n+\frac{1}{2}\right)t\,dt$$
$$= \Phi(\epsilon)\int_{\xi}^{\epsilon}\{g(x+t)-g(x)\}\sin\left(n+\frac{1}{2}\right)t\,dt = O\left(\frac{1}{n}\right).$$

Finally, we can replace here $\sin(n+\frac{1}{2})t$ by $\sin nt$ with error $O(1/n)$, and we have then that $\mathfrak{J}_2$ in (11) is equal to

$$\mathfrak{J}_2 = \int_{\phi(n)}^{\epsilon}\frac{g(x+t)-g(x)}{t}\sin nt\,dt + O\left(\frac{1}{n}\right).$$

If we choose $\phi(n)\to 0$ so that $[n\phi(n)]^{-1} = o(\omega(\pi/n))$ and if we apply the second mean value theorem to $1/t$ we obtain

$$\text{(12)}\qquad \begin{aligned}\mathfrak{J}_2 &= \frac{1}{\phi(n)}\int_{\phi_1(n)}^{\epsilon}[g(x+t)-g(x)]\sin nt\,dt + O\left(\frac{1}{n}\right)\\ &= O\left(\frac{1}{n\phi(n)}\right) = o(\omega(\pi/n)).\end{aligned}$$

Hence, for $t\to 0$, $g(x+t)-g(x)=Ct+L(t)$ and we have from (11)

$$\mathfrak{J}_1 = \int_0^{\phi(n)} \frac{L(t)}{t} \sin nt\, dt + O\left(\frac{1}{n}\right).$$

This integral is of the same type as $\mathfrak{J}_1$ in 2.1. It can be estimated in the same manner. We divide $\mathfrak{J}_1$ as in (4). Then we can choose δ small enough, respectively Δ sufficiently large, so that I_1 and I_3 in (5) and (6) are $o(L(\pi/n))$, and making $n\to\infty$ we have from (4)

$$\mathfrak{J}_1 = L\left(\frac{\pi}{n}\right) \int_\delta^{\Delta} \frac{\sin t}{t}\, dt + o\left(L\left(\frac{\pi}{n}\right)\right).$$

From this estimation and (12) we obtain by (11): $\mathfrak{J}=O(L(\pi/n))$. Thus Theorem II is proved.

References

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