ON EMBEDDING MANIFOLDS WHICH ARE BUNDLES OVER SPHERES[1]

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1. **Statement of results.** In this note we will prove the following theorem.

THEOREM 1.1. *Suppose M is the total space of a fiber bundle over S^n with fiber F, a compact, k-dimensional π-manifold. If F embeds in R^{k+m} with a normal field of n frames, then M topologically embeds in R^{2k+m+2} and differentiably embeds in R^{2k+m+j}, where $j \geqq 2$ and $j \geqq (m-k+3)/2$.*

The proof is given in §4. By a simplified version of the argument we also can prove

THEOREM 1.2. *Suppose M and F are as in Theorem 1.1. If F embeds in R^{k+q} with a trivial normal bundle for some $q \leqq n/2$, then M is topologically embeddable in R^{2k+n+1} and differentiably embeddable in R^{2k+n+j}, where $j \geqq 1$ and $j \geqq (n-k+3)/2$.*

The proof is given in §3. We have immediately this

COROLLARY 1.3. *If M is the total space of an S^k bundle over S^n, then M topologically embeds in R^{2k+n+1}.*

These results were discovered in an attempt to embed real projective spaces. The well-known Hopf fiberings of P_{15} and P_7 give the following representations of these spaces: P_7 is a fiber space over S^4 with P_3 as fiber, and P_{15} is a fiber space over S^8 with P_7 as fiber. Since any embedding of P_3 in R^5 has a trivial normal bundle [6], Theorem 1.2 applies for P_7. Theorem 1.1 applies for P_{15} and, hence, we have

THEOREM 1.4. *There exists a topological embedding of P_7 into R^{11}. There exists a differentiable embedding of P_7 into R^{12} and of P_{15} into R^{24}.*

COROLLARY 1.5. *There exists a differentiable embedding of P_{14} into R^{24}.*

In [5] it is proved that P_{14} embeds in R_{25} and P_{15} embeds in R^{28}. These results followed from rather extensive computations of secondary characteristic classes. The proof given here is geometric. James

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[3] has shown that P_{15} does not immerse in R_{21} and indicates that Sanderson has an immersion of P_{15} in R_{22}. It therefore seems likely that this embedding of P_{15} is the best possible. For other recent work on embedding projective spaces, see [1] and [2].

Another application is

THEOREM 1.6. SO(5) *embeds in* R^{17}.

PROOF. Since $SO(4) \cong SO(3) \times S^3$ [7, 8.6] and SO(5) is an SO(4) bundle over S^4, Theorem 1.2 applies.

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2. A lemma.

2.1. Let $g: F \times S^{n-1} \to F$ and let $Y = (F \times D^n) \cup_g F$. We can represent points in Y by triplets (f, s, r), where $f \in F$, $s \in S^{n-1}$ and $0 \leq r \leq 1$, under the equivalence relation $(f, s, r) \sim (f', s', r')$ iff

$$\begin{aligned} r = r' = 0 \quad &\text{and} \qquad f = f'; \text{ or} \\ r = r' = 1 \quad &\text{and} \quad g(f, s) = g(f', s'); \text{ or} \\ &(f, s, r) = (f', s', r'). \end{aligned} \tag{2.1.1}$$

Let $h_1: F \times D^n \to R^q$ and $h_2: F \to S^p \subset R^{p+1}$ be embeddings. An embedding of Y in R^{q+p+1} can be realized by joining $h_1(f, s, 1)$ and $h_2(g(f, s))$ with a line segment. This topological embedding is essentially given by

$$(f, s, r) \to ((1 - r)h_1(f, s, r), rh_2(g(f, s))). \tag{2.1.2}$$

This embedding is, in general, not very good since the full strength of h_2 is needed only when $r = 1$ and for this value of r, h_1 is not needed. Lemma 2.2 states a weaker condition under which we still get an embedding.

LEMMA 2.2. *Let F be a compact space and let D^n be the unit ball in R^n. Suppose there exists a map $I: F \times F \times D^n \to R^j$ such that:*

(i) *If $|x| = 0$ or 1, then $I(f_1, f_2, x)$ is independent of f_i, where $i = 1 + |x|$.*

(ii) *If $|x| = 1$, then I does not depend on x.*

(iii) *If $|x| < 1$, $|x| = |x'|$ and $I(f_1, f_2, x) = I(f_1', f_2', x')$, then $f_2 = f_2'$ and $x = x'$.*

(iv) *If $|x| = 1 = |x'|$ and $I(f_1, f_2, x) = I(f_1', f_2', x')$, then $f_1 = f_1'$. Then the total space M of any bundle over S^n with fiber F is topologically embeddable in R^{j+1}.*

Proof. Let s be any point in S^n and let $p: M \to S^n$ be the fiber map. Then $M - p^{-1}(s)$ is homeomorphic to $F \times R^n$. There exists a map $g: F \times S^{n-1} \to F$ such that $(F \times D^n) \cup_g F$ is homeomorphic to M, hence, we can represent M by ordered pairs (f, x) where $f \in F$ and $|x| < 1$ plus (f, x) where $|x| = 1$ modulo the equivalence relation $(f, x) \sim (f', x')$ iff $g(f, x) = g(f', x')$. Then the map $J: M \to R^j \times R^1$ defined by

$$J(f, x) = \left(I\left(g\left(f, \frac{x}{|x|} \right), f, x \right), \ |x| \right) \tag{2.2.1}$$

is the desired embedding. First observe that when $x = 0$, I is independent of f_1 by (i) and, hence, the apparent singularity in the first coordinate of the argument of I does not occur. Now suppose $J(f, x) = J(f', x')$. Then clearly $|x| = |x'|$. Suppose $|x| < 1$. By (iii) $f = f'$ and $x = x'$. If $|x| = 1$ then, by (iv), $g(f, x) = g(f', x')$ and this is just what is required for M.

3. **Proof of Theorem 1.2.** The hypotheses of Theorem 1.2 imply that there exist integers n_1 and n_2 and maps α_1 and α_2 such that $n_1 + n_2 = n$ and $\alpha_i: F \times R^{n_i} \to R^{k+n_i}$. Let points $x \in R^n$ be written as pairs (y, z) with $y \in R^{n_1}$ and $z \in R^{n_2}$. Let

$$I(f_1, f_2, (y, z)) = (r\alpha_1(f_1, (1 - r)y), (1 - r)\alpha_2(f_2, z)),$$

where $r = |(y, z)|$. Clearly I satisfies Lemma 2.1 and the topological portion of the theorem is proved.

The differentiable part follows from the topological part and this theorem of Haefliger [**4**]:

Theorem (Haefliger). *If there exists a topological embedding of an n-dimensional manifold M into R^k and if $2k \geqq 3(n+1)$, then there exists a differentiable embedding of M into R^k.*

4. **Proof of Theorem 1.1.**

4.1. We will first give a heuristic description of the proof. The details are given in 4.2. Since F is a π-manifold the normal bundle to any embedding of F in R^p, $p \geqq 2k+1$, will be trivial. In particular $F \times R^{k+m}$ embeds in R^{2k+m}. Using this we can get an embedding of $F \times F \times D^n$ into R^{2k+m}. Because the factor $F \times D^n$ is embedded in the normal bundle to the embedding of F we can not use a simple formula like 2.1.2 or even the idea of 3.1 to get an embedding of M. What we do is to construct a homotopy $h_t': F \times R^{k+m} \to R^{k+1} \times R^{k+m}$ based on a regular homotopy of an immersion of F in R^{k+1} into an embedding of F during which a normal field of $k+m$ frames is dragged along. This regular homotopy is then modified so that: (1) h_0' maps the second factor in a 1-1 fashion and maps the first factor trivially; (2) h_1' is an embedding of $F\times R^{k+m}$ into R^{2k+m+1}; (3) if $i_f\colon R^{k+m}\to F\times R^{k+m}$ is defined by $i_f(x)\to(f, x)$, then $h_t' i_f$ is an embedding for each f and each t. Let $\gamma\colon F\times D^n\to R^{k+m}$ be the embedding given by the hypothesis of the theorem. The map

$$(4.1.1)\qquad I(f_1, f_2, x) = h'_{|x|}(f_1, (1 - |x|)\gamma(f_2, x))$$

will satisfy Lemma 2.2.

4.2. Since F is a π-manifold, F can be immersed in R^{k+1}. Let α be such an immersion and let β be an embedding of F into R^{k+m}. Then $h_1\colon F\to R^{2k+m+1}$, defined by $h_1(f)=(\alpha(f), \beta(f))\in R^{k+1}\times R^{k+m}$, is an embedding of F into R^{2k+m+1}. Let $h_t\colon F\to R^{2k+m+1}$ be defined by $h_t(f)=(\alpha(f), (2t-1)\beta(f))$ for $\frac{1}{2}\leqq t\leqq 1$. This is a regular homotopy of the embedding h_1. Let $\hat{h}_t\colon F\to G_{k,k+m+1}$ be the map which assigns to each point in F the $k+m+1$ plane in R^{2k+m+1} which is normal to F at the point. Since h_t is a regular homotopy it is clear that $\hat{h}_t$ is a homotopy. Let $p\colon E\to G_{k,k+m+1}$ be the universal $V_{1,k+m}$ bundle.[2] (The space E can be thought of as the collection of pairs $\{$(a $k+m+1$ plane in R^{2k+m+1}, a $k+m$ frame in that $k+m+1$ plane)$\}$.) Let $\hat{H}_{1/2}\colon F\to E$ be the map which assigns to each point of F the $k+m$ frame in the normal $k+m+1$ plane consisting of the unit vectors formed by the last $k+m$ coordinates of R^{2k+m+1}. Clearly $p\hat{H}_{1/2}=\hat{h}_{1/2}$. By the *HCP* we can extend $\hat{H}_{1/2}$ to a homotopy $\hat{H}_t\colon F\to E$ which covers $\hat{h}_t$. Let $i\colon E\to V_{k+1,k+m}$ be the map which assigns to each $k+m$ frame in a $k+m+1$ subplane of R^{2k+m+1} the same frame considered as a $k+m$ frame in R^{2k+m+1}. Let $H_t=i\hat{H}_t$.

We can think of each point in $V_{k+1,k+m}$ as an ordered set of $k+m$ linearly independent vectors in R^{2k+m+1}. Let $H_t^i(f)$ be the ith vector of $H_t(f)$. Since F is compact we can choose H_t so that if $x\in D^{k+m}$, the unit disk in R^{k+m}, then $(f, x)\to\sum_{i=1}^{k+m} H_t^i(f)x_i+h_t(f)$ is an embedding of $F\times D^{k+m}$ for each $t>\frac{1}{2}$.

Clearly $H_{1/2}$ is a constant map and we define $H_t=H_{1/2}$ for $t\leqq\frac{1}{2}$. Let $h_t(f)=(2t\alpha(f), 0)$ for $t\leqq\frac{1}{2}$. Then

$$(4.2.1)\qquad \sum_{i=1}^{k+m} H_t^i(f)x_i + h_t(f) = (2t\alpha(f), x) \qquad \text{if } t\leqq\frac{1}{2}.$$

The homotopy $h_t'\colon F\times D^{k+m}\to R^{2k+m+1}$ is defined by

$$(4.2.2)\qquad h_t'(f, x) = \sum_{i=1}^{k+m} H_t^i(f)x_i + h_t(f).$$

[2] In this section we wish to consider $V_{k,n}$ as the collection of n frames in R^{k+n} which are not necessarily orthonormal.

Let γ: $F\times R^n \to R^{k+m}$ be the embedding whose existence follows from the embedding of F into R^{k+m} with a normal field of n-frames. Choose γ so that $|\gamma| \leqq 1$. Let $\gamma_i(f, x)$ be the ith coordinate of $\gamma(f, x)$. Let I: $F\times F\times D^n \to R^{2k+m+1}$ be defined by

$$(4.2.3)\quad I(f_1, f_2, x) = \sum_{i=1}^{m+k} (1 - |x|)H^i_{|x|}(f_1)\gamma_i(f_2, x) + h_{|x|}(f_1).$$

It is easily verified that I satisfies the conditions of Lemma 2.2. This proves the topological part. The differentiable part follows as in §3.

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