ON THE REDUCTION OF C^*-ALGEBRAS

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1. **Introduction.** Using the notion of the topological direct integral (cf. [3], [4]) M. Tomita has developed a reduction theory for C^*-algebras. Tomita has stated several theorems (cf. [5]) to the effect that a given C^*-algebra R can be written as a topological direct integral $R=\int_T \oplus R(t)d\nu(t)$ such that the component C^*-algebras $R(t)$ are irreducible for all $t\in T-N$, where N is a ν-local null set. Tomita's results seem to be incorrect, however, unless some separability conditions are imposed.[2] Theorems 1 and 2 in §2 credited to Tomita are correct if R is separable over its center (i.e., there exists a sequence of elements $\{A_n\}$ in R such that R is generated in the uniform topology by $\{A_n\}$ and the center of R). Using the theorems of Tomita, M. A. Naĭmark [3] has shown that every continuous unitary representation $g\rightarrow U_g$ of a locally compact group $\mathcal{G}$ is the topological direct integral $U_g=\int_T\oplus U_g(t)d\nu(t)$ of unitary representations $g\rightarrow U_g(t)$ which are nonzero, continuous, and irreducible for all $t\in T-N$, where N is a ν-local null set. For Naĭmark's theorem to be correct it must be assumed, however, by the preceding remarks, that $\mathcal{G}$ is separable. The purpose of this paper is to show that the exceptional null sets N in the theorems of Naĭmark and Tomita can be eliminated.

It is implicitly assumed that all locally compact groups and C^*-algebras considered in this paper are separable, respectively separable over their center.

We shall give a short résumé in the next section of the reduction theory for C^*-algebras stating only the theorems and results necessary for the understanding of this paper. For details we refer the reader to [2], [3] and [5].

2. **Preliminaries.** Let T be a locally compact Hausdorff space. With each t in T associate a Hilbert space $\mathcal{H}(t)$. A vector field defined on a subset D of T is a function x defined on D such that $x(t)\in\mathcal{H}(t)$

Received by the editors September 24, 1962 and, in revised form, April 10, 1963.

[1] National Science Foundation Fellow.

[2] Professor L. Loomis has pointed out to me the following error. In [3, p. 505] a reasoning depending on two-sided approximation is used in steps (c) and (d) where a one-sided approximation is only available. If R is separable in the uniform topology $\overline{N}$ has a countable basis and then the function x is itself a monotone limit of the sequence $\{y_n\}$ and the argument is correct by step (a).

for each $t \in D$. Let ν be a Radon measure on T whose support is T, and $C(T)$ the Banach space of all bounded continuous complex-valued functions on T. A *basis of a topological direct integral on* T is a family $\mathcal{S}$ of vector fields on T such that:

(i) for every x and y in $\mathcal{S}$ the inner product $(x(t) \mid y(t))$ is continuous on T and ν-integrable;

(ii) $\mathcal{S}$ is a vector space, i.e., if x and y are in $\mathcal{S}$ and α, β are complex numbers, then $z = \alpha x + \beta y \in \mathcal{S}$ where $z(t) = \alpha x(t) + \beta y(t)$;

(iii) if $x \in \mathcal{S}$ and $f \in C(T)$, then $y = fx \in \mathcal{S}$, where $y(t) = f(t)x(t)$;

(iv) for every $t \in T$ the set $\mathcal{S}(t) = \{x(t) \mid x \in \mathcal{S}\}$ is dense in $\mathcal{H}(t)$.

In $\mathcal{S}$ we introduce an inner product by the formula $(x \mid y) = \int_T (x(t) \mid y(t)) d\nu(t)$. Thus $\mathcal{S}$ becomes an inner product space.

A vector field x defined on $D \subset T$ is said to be continuous at the point t_0 of D (with respect to D), if for each $\epsilon > 0$ there exists a neighborhood U of t_0 in T and a $y \in \mathcal{S}$ such that $\|x(t) - y(t)\| < \epsilon$ for all $t \in U \cap D$.

A vector field x on T is said to be measurable (cf. [2]) (with respect to ν and $\mathcal{S}$) if for every $\epsilon > 0$ and every compact set $K \subset T$ there exists a compact set $K_1 \subset K$ such that x is continuous on K_1 (i.e., on every point of K_1 with respect to K_1) and $\nu(K - K_1) < \epsilon$.

Clearly, $\|x(t)\|$ is a ν-measurable function (in the sense of Bourbaki) if x is a measurable vector field. A vector field x is said to be square integrable if x is measurable and $\int_T \|x(t)\|^2 d\nu(t) < \infty$. The set of all square integrable vector fields is a complex vector space K. If $x \in K$, $y \in K$, $(x(t) \mid y(t))$ is a ν-integrable function. Denote the vector space K with the pseudo inner product $(x \mid y) = \int_T (x(t) \mid y(t)) d\nu(t)$ by $\mathcal{L}^2_{\mathcal{S}}(\nu)$. If $x \in \mathcal{L}^2_{\mathcal{S}}(\nu)$ denote by $\tilde{x}$ the class of all $y \in \mathcal{L}^2_{\mathcal{S}}(\nu)$ such that $x(t) = y(t)$, ν-almost everywhere. Finally denote by $L^2_{\mathcal{S}}(\nu)$ or ${}^{\mathcal{S}}\int_T \oplus \mathcal{H}(t) d\nu(t)$ or simply $\int_T \oplus \mathcal{H}(t) d\nu(t)$ the inner product space of equivalence classes $\tilde{x}$ with $(\tilde{x} \mid \tilde{y}) = \int_T (x(t) \mid y(t)) d\nu(t)$. If x and y are continuous vector fields in $\mathcal{L}^2_{\mathcal{S}}(\nu)$, $\tilde{x} = \tilde{y}$ if and only if $x = y$, since T is the support of ν. Hence we may identify the continuous square integrable vector fields x with $\tilde{x}$ and thus $\mathcal{S}$ becomes a subspace of $\int_T \oplus \mathcal{H}(t) d\nu(t)$. It is easy to see that $\int_T \oplus \mathcal{H}(t) d\nu(t)$ is a Hilbert space which contains $\mathcal{S}$ as a dense subspace (cf. [2]). $\int_T \oplus \mathcal{H}(t) d\nu(t)$ is called the (*topological*) *direct integral of the Hilbert spaces* $\mathcal{H}(t)$ (with respect to ν and $\mathcal{S}$). $\mathcal{S}$ is called the *basis* of the direct integral (cf. [3], [5]).

Suppose now that for every $t \in T$ is given a bounded operator $A(t)$ in $\mathcal{H}(t)$. If there exists a bounded operator A in ${}^{\mathcal{S}}\int_T \oplus \mathcal{H}(t) d\nu(t)$ such that $A\mathcal{S} \subset \mathcal{S}$ and $(Ax)(t) = A(t)x(t)$ for all $t \in T$, we say that A is *decomposable* and the *direct integral of the operators* $A(t)$ and write $A = \int_T \oplus A(t) d\nu(t)$ or simply $A = \int \oplus A(t)$. Note that the operators

$A(t)$ are uniquely determined by A because $\mathcal{S}(t)$ is dense in $\mathcal{H}(t)$ for every $t \in T$. If $A = \int \oplus A(t)$, then $\|A\| = \sup_{t \in T} \|A(t)\|$. If $A = \int \oplus A(t)$, $B = \int \oplus B(t)$, then $AB = \int \oplus A(t)B(t)$, $\alpha A + \beta B = \int \oplus (\alpha A(t) + \beta B(t))$. If $A = \int \oplus A(t)$ and $A^* = \int \oplus B(t)$, then $B(t) = A(t)^*$. If $f \in C(T)$, there exists an operator $C_f = \int \oplus f(t) I(t)$, where $I(t)$ is the identity operator in $\mathcal{H}(t)$. The mapping $f \rightarrow C_f$ is an isometric isomorphism. The image $\mathcal{C}$ of $C(T)$ under this isomorphism is called the *kernel algebra* of the direct integral. In what follows we shall often identify f with C_f.

If R is a C^*-algebra of decomposable operators $A = \int \oplus A(t)$ in $\mathcal{H} = \int_T \oplus \mathcal{H}(t) d\nu(t)$, then the mapping $A \rightarrow A(t)$ is for every $t \in T$ a $*$-algebraic homomorphism of R onto a C^*-algebra $R(t)$ of operators in $\mathcal{H}(t)$. This fact is denoted by writing $R = \int_T \oplus R(t) d\nu(t)$.

THEOREM 1 (TOMITA [5, *p.* 168]). *Let R be a C^*-algebra with identity on a Hilbert space $\mathcal{H}$ and cyclic vector x, $\|x\| = 1$. Then there exists an isometric isomorphism of $\mathcal{H}$ onto a topological direct integral ${}^{\mathcal{S}}\int_D \oplus \mathcal{H}(t) d\nu(t)$ such that*:

(a) *D is the spectrum of the kernel algebra of $\int_D \oplus \mathcal{H}(t) d\nu(t)$;*

(b) *if we identify $\mathcal{H}$ with $\int_D \oplus \mathcal{H}(t) d\nu(t)$, then $R = \int_D \oplus R(t) d\nu(t)$, where ν-almost all $R(t)$ are irreducible;*

(c) *$\mathcal{S}$ contains the cyclic vector x and for every $t \in T$, $x(t)$ is a cyclic vector for $R(t)$ and $\|x(t)\| = 1$;*

(d) *for every $t_1 \neq t_2$ in T there exists at least one $A \in R$ such that $(A(t_1)x(t_1) \mid x(t_1)) \neq (A(t_2)x(t_2) \mid x(t_2))$.*

If R_1 and R_2 are two C^*-algebras of operators on a Hilbert space $\mathcal{H}$ we denote by $R_1 \cup R_2$ the C^*-algebra generated by R_1 and R_2 in $L(\mathcal{H})$ (the Banach algebra of all bounded operators on $\mathcal{H}$). Let R be a C^*-algebra on $\mathcal{H}$. A commutative C^*-algebra of operators on $\mathcal{H}$ such that $(R \cup E)' = E$ is called *a diagonal algebra of R*. (If N is any set of operators in $\mathcal{H}$, then N' denotes the set of all bounded operators on $\mathcal{H}$ which together with their adjoints permute with every element in N.) Every C^*-algebra R has a diagonal algebra E. (Take E to be a maximal commutative self-adjoint subalgebra of R'.) We also note that a diagonal algebra E is always a W^*-algebra with identity since $E = (R \cup E)'$.

THEOREM 2 (TOMITA [5, *p.* 172]). *Let R be a C^*-algebra of operators on the Hilbert space $\mathcal{H}$ and E a diagonal algebra of R. Then there exists an isometric isomorphism of $\mathcal{H}$ onto a topological direct integral $\int_T \oplus \mathcal{H}(t) d\nu(t)$ such that if we identify $\mathcal{H}$ with $\int_T \oplus \mathcal{H}(t) d\nu(t)$, then*

(a) *the kernel algebra of the direct integral coincides with E, and*

(b) *$R = \int_T \oplus R(t) d\nu(t)$, where the $R(t)$ are irreducible for all $t \in T - N$ and N is a ν-local null set in T.*

3. The Main Theorems.

THEOREM 3. *Let R be a C^*-algebra with identity on a Hilbert space $\mathcal{H}$ and cyclic vector x, $\|x\|=1$, and E a diagonal algebra of R. Then there exists an isometric isomorphism of $\mathcal{H}$ onto a topological direct integral $^{\mathcal{S}}\int_T \oplus \mathcal{H}(t)d\nu(t)$ such that if we identify $\mathcal{H}$ with $\int_T \oplus \mathcal{H}(t)d\nu(t)$, then*

(a) *E is the diagonal algebra of $\int_T \oplus \mathcal{H}(t)d\nu(t)$;*

(b) *$R=\int_T \oplus R(t)d\nu(t)$, where the $R(t)$ are irreducible for all $t\in T$;*

(c) *$\mathcal{S}$ contains the cyclic vector x and for every $t\in T$, $x(t)$ is a cyclic vector for $R(t)$ and $\|x(t)\|=1$;*

(d) *for every $t_1\neq t_2$ in T there exists at least one $A\in R$ such that $(A(t_1)x(t_1)\,|\,x(t_1))\neq(A(t_2)x(t_2)\,|\,x(t_2))$.*

PROOF. $R\cup E$ is a cyclic C^*-algebra with cyclic vector x. Hence, by Theorem 1, there exists an isometric isomorphism of $\mathcal{H}$ onto $^{\mathcal{S}'}\int_D \oplus \mathcal{H}(t)d\rho(t)$ such that D is the spectrum of the kernel algebra of $\int_D \oplus \mathcal{H}(t)d\rho(t)$ and if we identify $\mathcal{H}$ with $^{\mathcal{S}'}\int_D \oplus \mathcal{H}(t)d\rho(t)$, $R\cup E =\int_D \oplus (R\cup E)(t)d\rho(t)$, where $(R\cup E)(t)$ is irreducible for all $t\in D-N$, and N is a ρ-null set. Using now exactly the same argument as in the proof of Theorem 2 ([5, pp. 171–172] or [3, p. 518]), we conclude that $(R\cup E)(t)=R(t)$ for all $t\in D$ and that E is the kernel algebra of $\int_D \oplus \mathcal{H}(t)d\rho(t)$.

Now, D as the spectrum of the W^*-algebra E with identity is a hyperstone space, and ρ is a normal measure on D whose support is D. Hence a subset M of D is nowhere dense if and only if M is a ρ-null set (cf. [1]). Therefore N is nowhere dense and hence $\overline{N}$ is a ρ-null set. Let $T=D-\overline{N}$ and $\mathcal{S}$ be the restriction of $\mathcal{S}'$ to T (i.e., $\mathcal{S}$ is the set of all vector fields defined on T which are restrictions of elements of $\mathcal{S}'$ to T). T is an open subset of D and hence a locally compact Hausdorff space. Every bounded continuous function f on T can be extended to a bounded continuous function f' on D (cf. [4, p. 215]), and since T is dense in D the extension f' is unique and $\sup_{t\in T}|f(t)| =\sup_{t\in D}|f'(t)|$. Hence $f\to f'$ is an isometric isomorphism of $C(T)$ onto $C(D)$ and if $x\in\mathcal{S}$, $f\in C(T)$, then $fx\in\mathcal{S}$. Finally, let ν be the restriction of the measure ρ to T. ν is a Radon measure on T whose support is T and $\mathcal{S}$ is a basis of a topological direct integral on T. The mapping $x'\to x$ of $\mathcal{S}'$ onto $\mathcal{S}$ (x is the restriction of x' to T) is an isometric isomorphism which can be uniquely extended to an isometric isomorphism of $^{\mathcal{S}'}\int_D \oplus \mathcal{H}(t)d\rho(t)$ onto $^{\mathcal{S}}\int_T \oplus \mathcal{H}(t)d\nu(t)$ and this isomorphism transforms R into $^{\mathcal{S}}\int_T \oplus R(t)d\nu(t)$ and E into the kernel algebra of $^{\mathcal{S}}\int_T \oplus \mathcal{H}(t)d\nu(t)$. The other statements are obvious.

COROLLARY 1. *Let R be a C^*-algebra with identity on a Hilbert space $\mathcal{H}$. Then there exists an isometric isomorphism of $\mathcal{H}$ onto a topological direct integral $^{\mathcal{S}}\int_T \oplus \mathcal{H}(t)d\nu(t)$ such that if we identify $\mathcal{H}$ with*

$\int_T \oplus \mathcal{H}(t)d\nu(t)$, then $R = \int_T \oplus R(t)d\nu(t)$, where the $R(t)$ are irreducible for all $t \in T$.

PROOF. The proof follows at once if we write $\mathcal{H}$ as an orthogonal sum of subspaces $\mathcal{H}_\alpha$ which are invariant with respect to R and each of which contains a cyclic vector x_α. Let R_α be the restriction of R to $\mathcal{H}_\alpha$ and apply the preceding theorem to $\mathcal{H}_\alpha$ and R_α. For details, cf. [3] or [5].

THEOREM 4. *Let R be a C^*-algebra with identity on a Hilbert space $\mathcal{H}$ and E a diagonal algebra of R. Then there exists an isometric isomorphism of $\mathcal{H}$ onto a topological direct integral $\int_T \oplus \mathcal{H}(t)d\nu(t)$ such that if we identify $\mathcal{H}$ with $\int_T \oplus \mathcal{H}(t)d\nu(t)$, then*

(a) *E is the kernel algebra of $\int_T \oplus \mathcal{H}(t)d\nu(t)$;*

(b) *$R = \int_T \oplus R(t)d\nu(t)$, where the $R(t)$ are irreducible for all $t \in T$.*

PROOF. Apply Corollary 1 to $R \cup E$. The rest of the proof is identical with the proof of Theorem 2 (cf. [5]).

REMARK. Corollary 1 and Theorem 4 remain true if R does not contain the identity operator I. For in that case consider the C^*-algebra R_1 obtained from R by adjoining I. Clearly E is a diagonal algebra of R_1, if it is a diagonal algebra of R. Apply now Corollary 1, respectively Theorem 4, to R_1. We obtain a direct integral decomposition $R_1 = \int_T \oplus R_1(t)d\nu(t)$, where all $R_1(t)$ are irreducible. Hence $R = \int_T \oplus R(t)d\nu(t)$. Clearly $R_1(t)$ is the C^*-algebra obtained from $R(t)$ by adjoining the identity operator $I(t)$ in $\mathcal{H}(t)$ and therefore $R_1(t)$ is irreducible for all $t \in T$.

THEOREM 5. *Let $g \rightarrow U_g$ be a continuous unitary representation of a locally compact group $\mathcal{G}$ on a Hilbert space $\mathcal{H}$. Then $\mathcal{H}$ can be realized in the form of a topological direct integral $\mathcal{H} = \int_T \oplus \mathcal{H}(t)d\nu(t)$ such that*

$$U_g = \int_T \oplus U_g(t)d\nu(t),$$

where $g \rightarrow U_g(t)$ is for all $t \in T$ a nonzero, irreducible, continuous unitary representation of $\mathcal{G}$.

PROOF. Without loss of generality we may assume that $g \rightarrow U_g$ is a cyclic representation (cf. [3]). Now proceed as in [3] using Theorem 3 instead of Theorem 1. It follows that there exists an isometric isomorphism of $\mathcal{H}$ onto a topological direct integral $\int_{D_1} \oplus \mathcal{H}(t)d\nu_1(t)$ such that if we identify $\mathcal{H}$ with $\int_{D_1} \oplus \mathcal{H}(t)d\nu_1(t)$ then $U_g = \int_{D_1} \oplus U_g(t)d\nu_1(t)$, and $g \rightarrow U_g(t)$ is a continuous irreducible nonzero unitary representation of $\mathcal{G}$ for all $t \in D_1 - N_2$, where N_2 is a closed subset of D_1 of ν_1-

measure zero. (Note that $N_2 = N_2'$ using the notation of [3].) Let $T = D_1 - N_2$ and ν the restriction of ν_1 to T. Using the same notations as in the proof of Theorem 3, we have that $D_1 = D - \overline{N}$ and ν_1 is the restriction of a normal measure ρ on the hyperstone space D with support D and $\rho(\overline{N}) = 0$. Hence $T = D - (\overline{N} \cup N_2)$ and ν is the restriction of ρ to T. N_2 is ρ-measurable, since it is ν_1-measurable. Hence $\rho(N_2) = \nu_1(N_2) = 0$ and therefore $\rho(\overline{N} \cup N_2) = 0$. Hence $\overline{N} \cup N_2$ is a closed ρ-null set of D_1 and the remainder of the proof is identical with the proof of Theorem 3.

COROLLARY 2. *Let p be a continuous positive definite function on a locally compact group $\mathcal{G}$ such that $p(e) = 1$. Then there exists an integral representation*

$$p(x) = \int_T p_t(x) d\nu(t),$$

where ν is a (bounded) Radon measure on T with support T and $p_t(x)$ is, for every $t \in T$, an elementary continuous positive definite function on $\mathcal{G}$. Furthermore $p_t(e) = 1$ for all $t \in T$ and $p_t(x)$ is a continuous function on T for every fixed $x \in \mathcal{G}$.

PROOF. $p(x) = (U_x a \mid a)$, where $x \to U_x$ is a continuous unitary representation of $\mathcal{G}$ on a Hilbert space $\mathcal{H}$ with cyclic vector a, $\|a\| = 1$. Hence

$$p(x) = (U_x a \mid a) = \int_T (U_x(t) a(t) \mid a(t)) d\nu(t)$$

$$= \int_T p_t(x) d\nu(t)$$

by (the proof of) Theorem 5.

THEOREM 6. *Let X be a separable compact Hausdorff space and μ a positive Radon measure on X invariant under a group $\mathcal{G}$ of automorphism of X. Then there exists a locally compact space T of positive $\mathcal{G}$-invariant bounded ergodic measures, and a (bounded) positive Radon measure m on T with support T, such that*

(a) $\mu = \int_T dm(\nu)$; *i.e.,*

$$\int_X f(x) d\mu(x) = \int_T dm(\nu) \int_X f(x) d\nu(x) \quad \text{for all } f \in C(X);$$

(b) *the mapping $\nu \to \int_X f(x) d\nu(x)$ is continuous on T for every $f \in (X)$.*

PROOF. (Cf. [2].) Let $\mathcal{H}=L^2(\mu)$ and $U_f, f\in C(X)$, and U_s, $s\in\mathcal{G}$, be the operators $U_fg=fg$ and $(U_sg)(x)=g(s(x))$ in $\mathcal{H}$, respectively. Then μ is ergodic if and only if the system of operators $\{U_f, U_s\}$ is irreducible (cf. [2]). Let R be the C^*-algebra generated by this system of operators. Then 1 is a cyclic vector for R and by Theorem 3 there exists an isometric isomorphism of $\mathcal{H}$ onto a topological direct integral ${}^{\mathcal{S}}\int_T\oplus\mathcal{H}(t)dm(t)$ which maps R into $\int_T\oplus R(t)dm(t)$ such that $R(t)$ is irreducible for all $t\in T$ and $1\in\mathcal{S}$. If $f\in C(X)$, then

$$\int_X f(x)d\mu(x) = (U_f1 \mid 1) = \int_T (U_f(t)1(t) \mid 1(t))dm(t).$$

The mapping $f\rightarrow(U_f(t)1(t)\mid 1(t))$ is clearly for every $t\in T$ a positive linear functional on $C(X)$; i.e., a positive Radon measure ν_t on X: $(U_f(t)1(t)\mid 1(t))=\int_X f(x)d\nu_t(x)$. Moreover $(U_f(t)1(t)\mid 1(t))$ is a continuous function on T since $1\in\mathcal{S}$ and $\nu_{t_1}\neq\nu_{t_2}$ for $t_1\neq t_2$ by (d) of Theorem 3. Now,

$$\int_X f(x)d\mu(x) = \int_X f(s(x))d\mu(x);$$

i.e.,

$$\int_T dm(t)\int_X f(x)d\nu_t(x) = \int_T dm(t)\int_X f(s(x))d\nu_t(x)$$

for $f\in C(X)$. Hence

$$\int_X f(x)d\nu_t(x) = \int_X f(s(x))d\nu_t(x) \quad \text{for all } t\in T$$

by continuity and the fact that T is the support of m. Finally ν_t is ergodic for every $t\in T$ because $R(t)$ is irreducible for every $t\in T$. This completes the proof.

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