References


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A-GENUS AND INDECOMPOSABILITY OF DIFFERENTIABLE MANIFOLDS

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Introduction. In the previous paper [1] we have studied the conditions on which a differentiable manifold be indecomposable and cited many examples of indecomposable manifolds. In this paper we shall study the relations between A-genus and indecomposability of a differentiable manifold.

1. Hereafter we denote by $X_n$ an $n$-dimensional compact orientable differentiable manifold. If $X_n = X_r \cdot X_s$ we say that $X_n$ is decomposable and if not, we say that $X_n$ is indecomposable. If $X_{4n} = X_r \cdot X_s$ we have

$$A(X_{4n}) = A(X_r)A(X_s);$$

where $A(X)$ denotes the $A$-genus of $X$ and we define as follows:

$$A(X_n) = 0, \quad n \not\equiv 0 \mod 4.$$

If $r$ and $s$ are divisible by 4, the relation (1.1) follows from the general property of multiplicative series [2, p. 75]. According to the cobordism theory, the cobordism components of $X_r$ ($r \not\equiv 0 \mod 4$) consist only of torsions. Hence the product $X_r \cdot X_s$ also consists only of torsions. Therefore $A(X_r \cdot X_s)$ is zero. Thus (1.1) holds in general. Meanwhile Atiyah and Hirzebruch have proved the following:

Theorem 1 (Atiyah and Hirzebruch [3]). If $X_{4n}$ is differentiably imbedded in the $(8n-2q)$-sphere, then $A(X_{4n})$ is divisible by $2^{s+1}$. If moreover $q \equiv 2 \mod 4$, then $A(X_{4n})$ is divisible by $2^{s+2}$.

It is well known that an $X_n$ is always differentiably imbedded in the $2n$-sphere. Hence we have from the above theorem

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(1.3) \( A(X_{4n}) \equiv 0 \mod 2 \).

We have from (1.1) and (1.3):

**Theorem 2.** If \( A(X_{4n}) \equiv 2 \mod 4 \), such an \( X_{4n} \) is indecomposable.

For example \( P_2(c) \) is indecomposable, because

\[
A(P_{2m}(c)) = (-1)^m \binom{2m}{m}
\]

(see [3]). We have proved the above theorem for the cases where \( n = 2, 3, 4 \). We shall find another example of Theorem 2 in the next paragraph.

2. Let \( P_{2m+1}(c) \) be the complex projective space of complex dimension \( 2m+1 \). The total Pontryagin class of \( P_{2m+1}(c) \) takes the form

\[
\phi = (1 + g^2)^{2m+2}, \quad g \in H^1(P_{2m+1}(c), Z).
\]

Let \( X_{4m} \) be a compact orientable differentiable submanifold of \( P_{2m+1}(c) \) and let \( \lambda g \) be the cohomology class corresponding to the homology class represented by \( X_{4m} \). The \( A \)-genus of \( X_{4m} \) is given by

\[
A(X_{4m}) = \prod_i \frac{2\sqrt{\nu_i}}{\sinh 2\sqrt{\nu_i}}
\]

where

\[
\phi(X_{4m}) = \prod_i (1 + \nu_i).
\]

Hence we have

\[
A(X_{4m}) = \prod_i \frac{2\sqrt{\nu_i}}{\sinh 2\sqrt{\nu_i}}[P_{2m+1}(c)] \quad [2, \text{p. 86}].
\]

Taking

\[
g^{2m+1}[P_{2m+1}(c)] = 1
\]

into account, we have

\[
A(X_{4m}) = \frac{1}{2\pi i} \int \frac{1}{2^{m+1} \zeta^{2m+2}} \left( \frac{2\lambda z}{\sinh 2\zeta} \right)^{2m+2} \, dz,
\]

where the integrand should be integrated along a small circle around \( z = 0 \) anti-clockwise. Changing the variable as follows:

\[
\sinh 2z = t
\]
we have

\[
A(X_\ell m) = \frac{2^{2m}}{2\pi i} \int \frac{\sinh 2\lambda z}{t^{2m+2}(1 + t^2)^{1/2}} dt
\]

(2.8)

where the integrand should be integrated along the small circle around \( t = 0 \) anti-clockwise. We have from (2.8):

\[
\begin{align*}
A(X_\ell m) &= \frac{2^{2m}}{2\pi i} \int \frac{1}{t^{2m+2}} \left\{ \lambda t(1 + t^2)^{(\lambda/2)-1} + \binom{\lambda}{3} t^3(1 + t^2)^{(\lambda/2)-2} \\
& \quad + \binom{\lambda}{5} t^5(1 + t^2)^{(\lambda/2)-3} + \cdots \right\} dt \\
&= 2^{2m} \left\{ \lambda \binom{\lambda/2 - 1}{m} + \binom{\lambda}{3} \binom{\lambda/2 - 2}{m-1} \\
& \quad + \binom{\lambda}{5} \binom{\lambda/2 - 3}{m-2} + \cdots + \binom{\lambda}{2m+1} \right\}
\end{align*}
\]

(2.9)

\[
= 2^{2m} \lambda \frac{(\lambda - 2) \cdots (\lambda - 2m)}{(2m + 1)!} \frac{1}{2m!} + \frac{\lambda - 1}{3 \cdot 2^{m-1} (m - 1)!} \\
+ \cdots + \frac{(\lambda - 1)(\lambda - 3) \cdots (\lambda - 2m + 1)}{(2m + 1)!}
\]

\[
= 2^{2m} \lambda f(\lambda).
\]

It is clear from (2.8) and (2.9) that

(2.10) \quad f(\lambda) = f(-\lambda)

and

(2.11) \quad f(2) = f(4) = \cdots = f(2m) = f(-2) \\
\quad = f(-4) = \cdots = f(-2m) = 0.

Therefore \( f(\lambda) \) takes the form

(2.12) \quad f(\lambda) = c(\lambda - 2)(\lambda - 4) \cdots (\lambda - 2m)(\lambda + 2)(\lambda + 4) \cdots (\lambda + 2m).

Letting \( \lambda = 1 \) in (2.9) we have

(2.13) \quad c = 1/2^{2m} \cdot 3 \cdot 5 \cdots (2m + 1).

Hence we have from (2.9), (2.12) and (2.13)
\[(2.14) \quad A(X_{4m}) = \frac{2^{m\lambda}}{m!} \prod_{i=1}^{m} \left( \frac{\lambda^2 - 4i^2}{2i + 1} \right). \]

If \( m = 2^s (s \geq 1) \) and \( \lambda \) is odd we have from (2.14)
\[(2.15) \quad A(X_{4m}) = 2 \mod 4. \]

Hence we have from Theorem 2

**Theorem 3.** Let \( X_{4m} \) be a submanifold of \( P_{2m+1}(\mathbb{C}) \) corresponding to \( \lambda g \). Then \( X_{4m} \) is indecomposable provided that \( m = 2^s (s \geq 1) \) and \( \lambda \) is odd.

Next we consider the case where \( w_2 = 0 \). It is known that if \( w_2 = 0 \), then \( A(X_{4n}) = 2^{4n} A(X_{4n}) \) is an integer, i.e., \( A(X_{4n}) \) is divisible by \( 2^{4n} \) [4]. We see from (2.14) that if \( w_2 = 0 \), then \( \lambda \) must be even. Thus we have

**Theorem 4.** If a submanifold \( X_{4n} \) of \( P_{2m+1}(\mathbb{C}) \) corresponding to \( \lambda g \), has the vanishing \( w_2 \), then \( \lambda \) is even.

**References**


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