A -GENUS AND INDECOMPOSABILITY OF DIFFERENTIABLE MANIFOLDS

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Introduction. In the previous paper [1] we have studied the conditions on which a differentiable manifold be indecomposable and cited many examples of indecomposable manifolds. In this paper we shall study the relations between A -genus and indecomposability of a differentiable manifold.

1. Hereafter we denote by \( X_n \) an \( n \)-dimensional compact orientable differentiable manifold. If \( X_n = X_r \cdot X_s \), we say that \( X_n \) is decomposable and if not, we say that \( X_n \) is indecomposable. If \( X_{4n} = X_n \cdot X_s \) we have

\[
A(X_{4n}) = A(X_r)A(X_s);
\]

where \( A(X) \) denotes the A -genus of \( X \) and we define as follows:

\[
A(X_n) = 0, \quad n \neq 0 \mod 4.
\]

If \( r \) and \( s \) are divisible by 4, the relation (1.1) follows from the general property of multiplicative series [2, p. 75]. According to the cobordism theory, the cobordism components of \( X_r \) (\( r \neq 0 \mod 4 \)) consist only of torsions. Hence the product \( X_r \cdot X_s \) also consists only of torsions. Therefore \( A(X_r \cdot X_s) \) is zero. Thus (1.1) holds in general. Meanwhile Atiyah and Hirzebruch have proved the following:

Theorem 1 (Atiyah and Hirzebruch [3]). If \( X_{4n} \) is differentiably imbedded in the \((8n-2q)\)-sphere, then \( A(X_{4n}) \) is divisible by \( 2^{s+1} \). If moreover \( q \equiv 2 \mod 4 \), then \( A(X_{4n}) \) is divisible by \( 2^{s+2} \).

It is well known that an \( X_n \) is always differentiably imbedded in the \( 2n \)-sphere. Hence we have from the above theorem

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(1.3) \[ A(X_{4n}) \equiv 0 \mod 2. \]

We have from (1.1) and (1.3):

**Theorem 2.** If \( A(X_{4n}) \equiv 2 \mod 4 \), such an \( X_{4n} \) is indecomposable.

For example \( P_{2m}(c) \) is indecomposable, because

\[ A(P_{2m}(c)) = (-1)^m \binom{2m}{m} \]

(see [3]). We have proved the above theorem for the cases where \( n = 2, 3, 4 \). We shall find another example of Theorem 2 in the next paragraph.

2. Let \( P_{2m+1}(c) \) be the complex projective space of complex dimension \( 2m+1 \). The total Pontryagin class of \( P_{2m+1}(c) \) takes the form

\[ (2.1) \quad p = (1 + g^2)^{2m+2}, \quad g \in H^2(P_{2m+1}(c), \mathbb{Z}). \]

Let \( X_{4m} \) be a compact orientable differentiable submanifold of \( P_{2m+1}(c) \) and let \( \lambda g \) be the cohomology class corresponding to the homology class represented by \( X_{4m} \). The \( A \)-genus of \( X_{4m} \) is given by

\[ (2.2) \quad A(X_{4m}) = \prod_i \frac{2\sqrt{\nu_i}}{\sinh 2\sqrt{\nu_i}} \]

where

\[ (2.3) \quad \rho(X_{4m}) = \prod_i (1 + \nu_i). \]

Hence we have

\[ (2.4) \quad A(X_{4m}) = \left[ \frac{\sinh 2\lambda g}{2} \left( \frac{2g}{\sinh 2g} \right)^{2m+2} \right] [P_{2m+1}(c)] \quad \text{[2, p. 86].} \]

Taking

\[ (2.5) \quad g^{2m+1}[P_{2m+1}(c)] = 1 \]

into account, we have

\[ (2.6) \quad A(X_{4m}) = \frac{1}{2\pi i} \int \frac{1}{z^{2m+2}} \left( \frac{\sinh 2\lambda z}{2} \right) \left( \frac{2z}{\sinh 2z} \right)^{2m+3} dz, \]

where the integrand should be integrated along a small circle around \( z = 0 \) anti-clockwise. Changing the variable as follows:

\[ (2.7) \quad \sinh 2z = t \]

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we have

\[ A(X_{4m}) = \frac{2^{2m}}{2\pi i} \int \frac{\sinh 2\lambda t}{t^{2m+2}(1 + t^2)^{1/2}} \, dt \]
\[ = \frac{2^{2m}}{2\pi i} \int \frac{((1 + t^2)^{1/2} + t)^\lambda - ((1 + t^2)^{1/2} - t)^\lambda}{2t^{2m+2}(1 + t^2)^{1/2}} \, dt \]

where the integrand should be integrated along the small circle around \( t = 0 \) anti-clockwise. We have from (2.8):

\[ A(X_{4m}) = \frac{2^{2m}}{2\pi i} \int \frac{1}{t^{2m+2}} \left\{ \lambda t(1 + t^2)^{\lambda/2-1} \left( \begin{array}{c} \lambda \\ 3 \end{array} \right) t^3(1 + t^2)^{\lambda/2-2} \\
+ \left( \begin{array}{c} \lambda \\ 5 \end{array} \right) t^5(1 + t^2)^{\lambda/2-3} + \cdots \right\} \, dt \]
\[ = 2^{2m} \left\{ \lambda \left( \frac{\lambda/2 - 1}{m} \right) + \left( \begin{array}{c} \lambda \\ 3 \end{array} \right) \frac{\lambda/2 - 2}{m - 1} \right. \\
+ \left. \left( \begin{array}{c} \lambda \\ 5 \end{array} \right) \frac{\lambda/2 - 3}{m - 2} + \cdots + \left( \begin{array}{c} \lambda \\ 2m + 1 \end{array} \right) \right\} \]
\[ = 2^{2m}\lambda(\lambda - 2) \cdots (\lambda - 2m) \left\{ \frac{1}{2^{2m}m!} + \frac{\lambda - 1}{3 \cdot 2^{2m-1}(m - 1)!} \right. \\
+ \left. \cdots + \frac{(\lambda - 1)(\lambda - 3) \cdots (\lambda - 2m + 1)}{(2m + 1)!} \right\} \]
\[ = 2^{2m}\lambda f(\lambda). \]

It is clear from (2.8) and (2.9) that

\[ f(\lambda) = f(-\lambda) \]

and

\[ f(2) = f(4) = \cdots = f(2m) = f(-2) \]
\[ = f(-4) = \cdots = f(-2m) = 0. \]

Therefore \( f(\lambda) \) takes the form

\[ f(\lambda) = c(\lambda - 2)(\lambda - 4) \cdots (\lambda - 2m)(\lambda + 2)(\lambda + 4) \cdots (\lambda + 2m). \]

Letting \( \lambda = 1 \) in (2.9) we have

\[ c = 1/2^{2m}m!3 \cdot 5 \cdots (2m + 1). \]

Hence we have from (2.9), (2.12) and (2.13)
If $m = 2^s (s \geq 1)$ and $\lambda$ is odd we have from (2.14)

\[(2.15) \quad \text{A}(X_{4m}) = 2 \mod 4.\]

Hence we have from Theorem 2

**Theorem 3.** Let $X_{4m}$ be a submanifold of $P_{2m+1}(c)$ corresponding to $\lambda g$. Then $X_{4m}$ is indecomposable provided that $m = 2^s (s \geq 1)$ and $\lambda$ is odd.

Next we consider the case where $w_2 = 0$. It is known that if $w_2 = 0$, then $\text{A}(X_{4m}) = 2^{4n} \text{A}(X_{4m})$ is an integer, i.e., $\text{A}(X_{4m})$ is divisible by $2^{4n}$ [4]. We see from (2.14) that if $w_2 = 0$, then $\lambda$ must be even. Thus we have

**Theorem 4.** If a submanifold $X_{4n}$ of $P_{2m+1}(c)$ corresponding to $\lambda g$ has the vanishing $w_2$, then $\lambda$ is even.

**References**


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