ARITHMETICAL NOTES. X. A CLASS OF TOTIENTS[1]

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1. **Introduction.** We shall call a divisor d of the positive integer n *unitary*, denoted $d\|n$ or $d * \delta = n$, provided $(d, \delta) = 1$ where $\delta = n/d$. The number n will be termed *k-free* if it has no prime factor of multiplicity $\geqq k$ and *k-full* if it has no prime factor of multiplicity $< k$. The sets of k-free and k-full integers will be denoted Q_k and L_k, respectively, and their characteristic functions $q_k(n)$ and $l_k(n)$. For simplicity, we place $Q = Q_2$, $L = L_2$, $q = q_2$, $l = l_2$.

It is recalled that the Moebius function $\mu(n)$ is the multiplicative function defined for primes p and integers $e \geqq 1$ by

$$\mu(p^e) = \begin{cases} -1 & \text{if } e = 1, \\ 0 & \text{if } e \geqq 2. \end{cases}$$

Equivalently, μ may be characterized by the relation,

$$\sum_{d|n} \mu(d) = \epsilon(n) \overset{\text{def}}{=} \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \neq 1. \end{cases} \tag{1}$$

From the discussion of §2 it follows that μ can also be characterized by a relation involving unitary divisors, namely,

$$\sum_{d\|n} \mu(d) = l(n) = \begin{cases} 1 & \text{if } n \in L, \\ 0 & \text{if } n \notin L, \end{cases} \tag{2}$$

or by the alternate formulation,

$$\sum_{d*\delta=n} \mu(d) q(\delta) = \epsilon(n). \tag{2a}$$

It is a point of interest that the "unitary" characterization (2), rather than the classical relation (1), generalizes most naturally with respect to the generalized Moebius function μ_k^*, defined to be the multiplicative function such that

$$\mu^*(p^e) = \begin{cases} -1 & \text{if } 1 \leqq e < k, \\ 0 & \text{if } e \geqq k. \end{cases} \tag{3}$$

Received by the editors August 27, 1962.

[1] This research was supported in part by the National Science Foundation (G-19012).

Evidently, $\mu=\mu_2^*$, and (2) and (2a) result from the case $k=2$ of Theorem 1.

Let $(m, n)_*$ denote the largest common divisor of m and n which is a unitary divisor of n. We shall say that m is *semi-prime* to n if $(m, n)_*=1$. Define $\Phi_k^*(n)$ to be the number of integers m in a residue system (mod n) which are semi-prime to the maximal unitary divisor $\alpha_k(n)$ of n contained in Q_k (see Remark 1 below). In Theorem 3, a simple representation of Φ_k^* in terms of the generalized Moebius function μ_k^* is given. This representation is then used in §3 to prove in an elementary way the asymptotic expression for the average of Φ_k^* contained in Theorem 4. For another application of the function μ_k^* the reader is referred to [5].

The function $\Phi_k^*(n)$ may be viewed as a unitary analogue of the function $\Phi_k(n)$, defined to be the number of m (mod n) prime to $\alpha_k(n)$. An asymptotic expression for the average of Φ_k, analogous to that obtained in this note for Φ_k^*, was proved in [3, Theorem 3b]. The proof of the latter result was not, however, strictly elementary because it involved the use of generating functions based on real Dirichlet series. On the basis of the observation that $\Phi_2(n)=\Phi_2^*(n)$, the case $k=2$ of Theorem 4 of this paper (see Corollary 1) furnishes a new and elementary proof of a result proved in [3, (4.6)]. This case is of further interest by reason of its relation to the Moebius function (see Corollary 1 of Theorem 3).

REMARK 1. Note that every n has a unique decomposition of the form $n=d * \delta$ where $d\in L_k$, $\delta\in Q_k$; in particular, $\delta=\alpha_k(n)$.

REMARK 2. The notation for Φ_k in [3] differs slightly from that used here.

2. **Formal relations.** Let us denote by $h=f * g$ the unitary product of (complex-valued) arithmetical functions f, g, defined by

$$h(n) = \sum_{d*\delta=n} f(d)g(\delta).$$

LEMMA 1. *Relative to the operations of function addition and unitary multiplication, the set of all arithmetical functions forms a commutative ring R with identity ϵ; under the unitary product the multiplicative functions form a cancellation semigroup G with identity ϵ.*

PROOF. Except for the cancellation property of G, this result is contained in [2, Theorem 2.1], (also cf. [4, Lemma 2]). Let g denote a function of G and f a function of R such that $f * g=z$, where z is the zero element of R, that is, $z(n)=0$ for all n. It must follow that $f=z$, because otherwise there exists a smallest positive integer n_1 such that

$f(n_1) \neq 0$, from which one deduces that $(f * g)(n_1) = f(n_1)$. Hence g is a regular element of R, and since the regular elements of a commutative ring form a cancellation semigroup under multiplication, the proof is complete.

Let ρ be the function of G defined by $\rho(n) = 1$ for all n.

THEOREM 1. *For all* $k \geqq 1$

$$\sum_{d \| n} \overset{*}{\mu}_k(d) = l_k(n); \tag{4}$$

equivalently,

$$\sum_{d * \delta = n} \overset{*}{\mu}_k(d) q_k(\delta) = \epsilon(n). \tag{4a}$$

REMARK 3. The relation (4) can be restated in the form $\mu_k^* * \rho = l_k$.

PROOF. Since ρ, μ_k^*, and l_k are all contained in G, it suffices by Lemma 1 to verify (4) in the case $n = p^e$. In this case the left of (4) becomes $1 + \mu_k^*(p^e) = 0$ if $e < k$, 1 if $e \geqq k$, which proves (4). By Remark 1, $l_k * q_k = \rho$; hence by the first part of Lemma 1 and Remark 3,

$$(\overset{*}{\mu}_k * q_k) * \rho = l_k * q_k = \epsilon * \rho,$$

and (4a) results by the cancellation property of G (second part of Lemma 1).

The relation $\mu_k^* * q_k = \epsilon$ of (4a) asserts that μ_k^* and q_k are elements of the multiplicative group of G and are inverses of each other, $\mu_k^* = q_k^{-1}$. Hence $g = f * q_k \rightleftarrows f = g * \mu_k$, which can be restated as the inversion formula:

THEOREM 2. *Let f and g be functions of R; then*

$$g(n) = \sum_{d * \delta = n} f(d) q_k(\delta) \rightleftarrows f(n) = \sum_{d * \delta = n} g(d) \overset{*}{\mu}_k(\delta).$$

The case $k = 2$ of Theorem 2 yields an inversion formula involving the Moebius function.

COROLLARY 1 ($k = 2$).

$$g(n) = \sum_{d * \delta = n} f(d) q(\delta) \rightleftarrows f(n) = \sum_{d * \delta = n} g(d) \mu(\delta).$$

Define the function μ^* by $\mu^*(n) = \lim_{k \to \infty} \mu_k^*(n)$. Then we obtain the unitary analogue of the classical Moebius inversion formula [**1**, Theorem 2.3].

COROLLARY 2 ($k \to \infty$).

$$g(n) = \sum_{d*\delta=n} f(d) \rightleftarrows f(n) = \sum_{d*\delta=n} g(d)\mu^*(\delta).$$

THEOREM 3.

$$\bar{\phi}_k^*(n) = \sum_{d*\delta=n} d\mu_k^*(\delta); \tag{5}$$

equivalently,

$$\sum_{d*\delta=n} \bar{\phi}_k^*(d) q_k(\delta) = n. \tag{5a}$$

PROOF. The formulas (5) and (5a) are equivalent by Theorem 2. It therefore suffices to prove (5a). To do this let δ denote a unitary divisor of $\alpha_k(n)$. The number of m in a complete residue system (mod n) such that $(m, \alpha_k(n))_* = \delta$ is $\bar{\phi}_k^*(n/\delta)$; hence

$$n = \sum_{\delta \| \alpha_k(n)} \bar{\phi}_k^*(n/\delta),$$

but this is merely a reformulation of (5a). Q.E.D.

Place $\bar{\phi}^*(n) = \bar{\phi}_2^*(n)$.

COROLLARY 1 ($k=2$).

$$\bar{\phi}^*(n) = \sum_{d*\delta=n} d\mu(\delta), \qquad n = \sum_{d*\delta=n} \phi(d)q(\delta).$$

Define $\phi^*(n)$ to be $\lim_{k\to\infty} \bar{\phi}_k^*(n)$. Then we have [**1**, (2.4), (2.8)]:

COROLLARY 2 ($k\to\infty$).

$$\phi^*(n) = \sum_{d*\delta=n} d\mu^*(\delta), \qquad n = \sum_{d\|n} \phi^*(d).$$

3. **Asymptotic expressions.** We first note that

LEMMA 2. *For all* $k \geqq 1$, $|\mu_k^*(n)| \leqq 1$.

Let $\phi(n)$ denote the classical ϕ-function.

LEMMA 3. *For all* $k \geqq 2$,

$$\sum_{n=1}^{\infty} \frac{\mu_k(n)\phi(n)}{n^3} = c_k \stackrel{\text{def}}{=} \prod_p \left\{1 - \frac{1}{p^{2k-1}}\left(\frac{p^{2k-2}-1}{p+1}\right)\right\}, \tag{6}$$

where the product is extended over all primes p.

PROOF. The series in (6) is absolutely convergent, by Lemma 2 and the fact that $\phi(n) \leqq n$. Hence by (3), on taking Euler products and recalling that $\phi(p^n) = p^{n-1}(p-1)$,

$$\sum_{n=1}^{\infty} \frac{\overset{*}{\mu}_k(n)\phi(n)}{n^3} = \prod_p \left(1 - \sum_{j=1}^{k-1} \frac{\phi(p^j)}{p^{3j}}\right) = \prod_p \left\{1 - \left(\frac{p-1}{p^3}\right) \sum_{j=0}^{k-2} \frac{1}{p^{2j}}\right\}$$
$$= \prod_p \left\{1 - \left(\frac{p-1}{p^3}\right)\left(\frac{1 - p^{-2(k-1)}}{1 - p^{-2}}\right)\right\},$$

which is the same as (6).

Place $c = c_2$.

Corollary ($k=2$).

$$\sum_{n=1}^{\infty} \frac{\mu(n)\phi(n)}{n^3} = c = \prod_p \left(1 - \frac{1}{p^2} + \frac{1}{p^3}\right). \tag{7}$$

We shall need the following result proved in [1, Lemma 4.1].

Let x denote a real variable, $x \geqq 1$, and let $\theta(n)$ be the number of unitary divisors of n.

Lemma 4.

$$U(x, r) =^{\text{def}} \sum_{n \leqq x;\, (n,r)=1} n = \frac{x^2}{2}\left(\frac{\phi(r)}{r}\right) + O(x\theta(r)). \tag{8}$$

We are now ready to prove

Theorem 4. *If c_k is the product defined by* (6), *then for* $k \geqq 2$

$$\overset{*}{\Phi}_k(x) =^{\text{def}} \sum_{n \leqq x} \bar{\overset{*}{\phi}}_k(n) = c_k x^2/2 + O(x \log^2 x), \tag{9}$$

uniformly in k.

Proof. By Theorem 3, Lemma 2, Lemma 4, and the boundedness of $\phi(n)/n$,

$$\overset{*}{\Phi}_k(x) = \sum_{d\delta \leqq x;\, (d,\delta)=1} \overset{*}{\mu}_k(\delta) d = \sum_{n \leqq x} \overset{*}{\mu}_k(n) U\left(\frac{x}{n}, n\right)$$
$$= \frac{x^2}{2} \sum_{n=1}^{\infty} \frac{\overset{*}{\mu}_k(n)\phi(n)}{n^3} + O\left(x^2 \sum_{n>x} \frac{1}{n^2}\right) + O\left(x \sum_{n \leqq x} \frac{\theta(n)}{n}\right).$$

The first O-term is $O(x)$ and the second $O(x \log^2 x)$, (cf. [1, p. 73]). The theorem results on the basis of Lemma 3.

Corollary 1 ($k=2$).

$$\sum_{n \leqq x} \bar{\phi}^*(n) = \frac{cx^2}{2} + O(x \log^2 x), \tag{10}$$

where c is determined by (7).

The limiting case in (10) yields the result on the average order of $\phi^*(n)$ proved in [1, Corollary 4.1.2].

COROLLARY 2 ($k \rightarrow \infty$).

$$\Phi^*(x) = {}^{\mathrm{def}}\sum_{n\leq x} \phi^*(n) = \frac{\alpha x^2}{2} + O(x \log^2 x), \tag{11}$$

where

$$\alpha = \prod_p \left(1 - \frac{1}{p(p+1)}\right).$$

PROOF. By the uniform convergence of the product in (6), one obtains $\lim_{k\to\infty} c_k = \alpha$. Since the theorem holds uniformly in k, (11) results from (9) on passing to the limit.

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