HOMOGENEOUS ALMOST COMPLEX SPACES OF POSITIVE CHARACTERISTIC

SAMUEL PASIENCIER

In this paper we give a necessary and sufficient condition for a homogeneous space of positive Euler characteristic to admit an invariant almost complex structure (i.a.c.s.), Theorem 4.1. A classification of these spaces has already been accomplished by R. Hermann [3], but no characterization of them has been given, to date. This work is a portion of the author's dissertation directed by H. C. Wang.

1. Let G be a compact, connected Lie group, and L a closed subgroup of G, $\mathfrak{G}$ and $\mathfrak{L}$ their Lie algebras. Then as is well known, L is reductive in G and we may write $\mathfrak{G}=\mathfrak{L}+\mathfrak{M}$, where $\operatorname{Ad} L\colon \mathfrak{M}\to\mathfrak{M}$. Moreover by homogeneity, a necessary and sufficient condition that there exist an almost complex structure on G/L, is that there exist a linear transformation $J\colon \mathfrak{M}\to\mathfrak{M}$, such that $J^2=-I$ (I, the identity), and such that $\operatorname{Ad} lJ=J \operatorname{Ad} l$ for all $l\in L$. See for example, A. Frölicher [2].

2. We give first a sufficient condition in the general case.

THEOREM 2.1. *Let Z be a closed, abelian subgroup of a compact Lie group G, and L the identity component of the centralizer of Z in G. If each 1-dimensional subspace of $\mathfrak{G}$ invariant by* Ad Z *is pointwise fixed, then G/L has an i.a.c.s.*

PROOF. Z being compact and abelian, Ad Z is completely reducible; with notation as in 1, there cannot exist any 1-dimensional subspaces of $\mathfrak{M}$ reducing Ad Z, for if $X\neq 0$ belonged to such a subspace, then by hypothesis we would have $\operatorname{Ad} z(X)=X$ for all $z\in Z$, and thus $X\in\mathfrak{L}$, which is impossible.

If $C\mathfrak{M}$ denotes the complexification of $\mathfrak{M}$, and Ad Z and Ad L are extended by linearity to $C\mathfrak{M}$, then we know by Schur's Lemma that Ad Z acting on $C\mathfrak{M}$ has a weight vector v_i. That is, $\operatorname{Ad} z(v_i)=\psi_i(z)v_i$ for all $z\in Z$, ψ_i being a character on Z. Moreover, since Ad Z is real, $\operatorname{Ad} z(\bar{v}_i)=\bar{\psi}_i(z)\bar{v}_i$, thus the weight vectors and roots appear in conjugate pairs.

Let $v_1, \cdots, v_n, \bar{v}_1, \cdots, \bar{v}_n$ be the weight vectors with characters $\psi_1, \cdots, \psi_n, \bar{\psi}_1, \cdots, \bar{\psi}_n$, where some of the ψ_i's may be identical. If

Received by the editors September 4, 1962 and, in revised forms, December 3, 1962 and April 9, 1963.

we let $\mathcal{V}$ be the subspace of $C\mathfrak{M}$ spanned by the v_i's, then each $x\in\mathfrak{M}$ has a unique expression of the form $x=v+\bar{v}$, where $v\in\mathcal{V}$ and $\bar{v}\in\bar{\mathcal{V}}$. For putting $x_j=v_j+\bar{v}_j$, and $x_j'=iv_j-i\bar{v}_j$ the x_j's and x_j''s span the irreducible subspaces of $\mathfrak{M}$. And thus any $x\in\mathfrak{M}$ has a unique expression

$$x = \sum a_j x_j + \sum b_j x_j',$$

so

$$x = \sum \lambda_j v_j + \sum \bar{\lambda}_j \bar{v}_j.$$

Furthermore, $\mathcal{V}$ is invariant by Ad L, for if $z\in Z$, $l\in L$, then Ad z Ad $l(v_i)=$ Ad l Ad $z(v_i)=\psi_i(z)$ Ad $l(v_i)$, that is, Ad $l(v_i)$ belongs to the character ψ_i, and by hypothesis there can be no real characters, thus $\psi_i\neq\bar{\psi}_j$. Now defining $J\colon\mathfrak{M}\to\mathfrak{M}$ by $J(x)=J(v+\bar{v})=iv-i\bar{v}$, defines an i.a.c.s. on G/L.

COROLLARY 2.1. *If the number of connected components in* Ad Z *is odd, then G/L has an i.a.c.s., G, L, Z as above.*

PROOF. Suppose there exists a 1-dimensional subspace of $\mathfrak{M}$, with basis X, reducing Ad Z, then for each $z\in Z$, Ad $z(X)=\psi(z)X$, where $\psi(z)$ is a homomorphism of $Z\to Z_2=\{+1, -1\}$. By continuty ψ is constant on components of Ad z, and takes the value $+1$ on the identity component $(\text{Ad } z)^0$. It thus induces a homomorphism $\psi\colon \text{Ad } z/(\text{Ad } z)^0\to Z_2$, but Ad $z/(\text{Ad } z)^0$ is a finite group of odd order, therefore its image is identically 1, so that $\psi\equiv 1$, and thus Ad z is pointwise fixed on each 1-dimensional subspace reducing it.

3. Next we consider the case that L is semi-simple.

LEMMA 3.1. *Let L, H be two closed subgroups of G, $L\subset H\subset G$. And suppose that G/L has an i.a.c.s. defined by $J\colon\mathfrak{M}\to\mathfrak{M}$, where $\mathfrak{G}=\mathfrak{L}+\mathfrak{M}$. If further $\mathfrak{H}=\mathfrak{L}+\mathfrak{M}_1$, $\mathfrak{M}_1\subset\mathfrak{M}$, and $J(\mathfrak{M}_1)=\mathfrak{M}_1$, then H/L has an i.a.c.s.*

PROOF. Since H is a subgroup of G, Ad $L\colon\mathfrak{M}_1\to\mathfrak{M}_1$. Thus applying 1, we see that H/L has an i.a.c.s.

LEMMA 3.2. *Let L be a closed semi-simple subgroup of G, of maximal rank, such that G/L has an i.a.c.s.; then L is not a symmetric subgroup.*

PROOF. Let $\mathfrak{G}=\mathfrak{L}+\mathfrak{M}$, and $J\colon\mathfrak{M}\to\mathfrak{M}$ define the i.a.c.s. as in §1. Since the torsion tensor of J is given by

$$\mathfrak{T}(X, Y) = [JX, JY] - J[JX, Y] - J[X, JY] + [X, Y] \bmod \mathfrak{L}$$

for X, $Y\in\mathfrak{M}$ [2], if we assume G/L is symmetric $\mathfrak{T}(X, Y)\equiv 0$, for

symmetric implies $[\mathfrak{M}, \mathfrak{M}] \subset L$. Thus J defines a complex structure on G_0/L_0 (where subscripts denote identity components), but then a classical result of E. Cartan says that L_0 (and thus L) cannot be semi-simple.

LEMMA 3.3. *If G/L has an i.a.c.s. and L is connected semi-simple of maximal rank, Z the center of L; then* $\mathrm{Ad}(Z)$ *acting on $\mathfrak{M}$ has no* 1-*dimensional invariant subspaces.*

PROOF. Suppose by contradiction that $\mathfrak{R} \subset \mathfrak{M}$ is 1-dimensional and reduces $\mathrm{Ad}(Z)$. Let $x \in \mathfrak{R}$, $x \neq 0$, then

$$\mathrm{Ad}\, z(x) = \psi(z)x,\ z \in Z, \text{ and } \psi \text{ is a real character on } Z.$$

Since Z is compact, either $\psi(Z) \equiv 1$, or $\psi(Z) = \{+1, -1\}$. From Borel-deSiebenthal [1], we know that L is the connected component of the centralizer of its own center Z, and thus $\mathfrak{L} = \{u : u \in \mathfrak{G}, \mathrm{Ad}\, Zu = u\}$. If $\psi(Z) \equiv 1$, then $\mathfrak{R} \subset \mathfrak{L}$, which is impossible. Thus $\psi(Z) = \{1, -1\}$.

Let $\mathfrak{Q} = \{y : y \in \mathfrak{M}, \mathrm{Ad}\, z(y) = \psi(z)y, z \in Z\}$ be the weight space corresponding to the character ψ. Then $J(\mathfrak{Q}) = \mathfrak{Q}$, for if $y \in \mathfrak{Q}$,

$$\mathrm{Ad}\, z\, J(y) = J \cdot \mathrm{Ad}\, z(y) = J\psi(z)y = \psi(z)Jy.$$

Furthermore, $\mathrm{Ad}\, L(\mathfrak{Q}) = \mathfrak{Q}$, for if $y \in \mathfrak{Q}$, $l \in L$, then

$$\mathrm{Ad}\, z \mathrm{Ad}\, ly = (\mathrm{Ad}\, l)\psi(z)y = \psi(z)\mathrm{Ad}\, ly.$$

But then

$$[\mathfrak{L}, \mathfrak{Q}] \subset \mathfrak{Q}.$$

Finally if $y, y' \in \mathfrak{Q}$, $\mathrm{Ad}\, z[y, y'] = \psi(z)^2[y, y'] = [y, y']$, that is, $[y, y']$ is fixed under all $\mathrm{Ad}\, z$, so that

$$[y, y'] \in \mathfrak{L}, \text{ that is } [\mathfrak{Q}, \mathfrak{Q}] \subset \mathfrak{L}.$$

Therefore, $\mathfrak{H} = \mathfrak{L} + \mathfrak{Q}$ is a subalgebra and H/L is a symmetric space, $H = \exp \mathfrak{H}$. Since H is of maximal rank, H is closed by the theorem in §8 of [1]. By Lemma 3.1, H/L has an invariant almost complex structure, and H/L is nontrivial since $\mathfrak{Q} \neq 0$, for $\mathfrak{R} \subset \mathfrak{Q}$. This contradicts Lemma 3.2, thus Lemma 3.3 is proven.

LEMMA 3.4. *Under the hypothesis of Lemma* 3.3, $\mathrm{Ad}(Z)$ *cannot be a group of order* 2^k, $k > 0$.

PROOF. (a) If $k = 1$, G/L is symmetric and this is impossible by Lemma 3.2.

(b) Suppose the lemma is true for all $k \leqq n$, and consider the case that $\mathrm{Ad}\, Z$ is of order 2^{n+1}. Since $\mathrm{Ad}\, Z$ contains a subgroup of order 2,

there exists an element $z_0 \in Z$ such that $\mathrm{Ad}\, z_0 \neq 1$ and $(\mathrm{Ad}\, z_0)^2 = 1$. Let G^* be the 1-component of the centralizer of z_0 in G. Then $L \subset G^*$, and we have

$$\mathfrak{G} = \mathfrak{L} + \mathfrak{M}, \qquad \mathfrak{M} = \mathfrak{Q}_1 + \mathfrak{Q}_2, \qquad \mathfrak{G}^* = \mathfrak{L} + \mathfrak{Q}_1,$$

where $\mathrm{Ad}\, z_0|_{\mathfrak{Q}_1} = 1$ and $\mathrm{Ad}\, z_0|_{\mathfrak{Q}_2} = -1$, $\mathfrak{G}^*$ being the Lie algebra of G^*. Moreover $J(\mathfrak{Q}_1) = \mathfrak{Q}_1$ for if $m \in \mathfrak{Q}_1$, $(\mathrm{Ad}\, z)J(m) = J(\mathrm{Ad}\, z)m = J(m)$. Similarly we show that $\mathrm{Ad}\, L(\mathfrak{Q}_1) = \mathfrak{Q}_1$.

G^*/L has i.a.c.s. by Lemma 3.1. But since $L \subset G^*$, Z is the center of L in G^*. Since $\mathrm{Ad}\, z_0|_{\mathfrak{G}^*} = 1$, $\mathrm{Ad}\, Z|_{\mathfrak{G}^*}$ is of order 2^m, $m \leqq n$. Thus by the inductive hypothesis $m = 0$ and $G^* = L$. But then G/L is a symmetric space, contradicting Lemma 3.2. Thus G/L cannot have invariant almost complex structure, and the lemma is proved.

THEOREM 3.5. *Let G be a compact connected Lie group and L a connected semi-simple subgroup of G of maximal rank; then G/L has invariant almost complex structure if and only if L is the identity component of the centralizer in G of a finite subgroup F, of its center, such that* $\mathrm{Ad}(F)$ *is of odd order.*

PROOF. The sufficiency of this condition was proved in Theorem 2.1. To prove the necessity, let Z be the center of L; then by the fundamental theorem of Abelian groups we may write $\mathrm{Ad}\, Z = \mathrm{Ad}\, F_1 \times \mathrm{Ad}\, F_2$, where $\mathrm{Ad}\, F_1$ is of odd order and $\mathrm{Ad}\, F_2$ is of order 2^k.

Let G_1 be the identity component of the centralizer of F_1 in G. We shall show that $G_1 = L$. Suppose on the contrary that $L \subsetneqq G_1$. Then G_1/L has i.a.c.s. by Lemma 3.1. If $\mathrm{Ad}(F_2): \mathfrak{G}_1 \to \mathfrak{G}_1$ is identically 1, then we are done because $\mathfrak{G}_1$ centralizes both F_1 and F_2 and by the result of Borel and de Siebenthal [1], $G_1 = L$. But $\mathrm{Ad}(F_1)$ is identically 1 on $\mathfrak{G}_1$, and $\mathrm{Ad}(F_2)$, if nontrivial on $\mathfrak{G}_1$, must be of order 2^n, for some $n > 1$. But then $\mathrm{Ad}\, Z$ acting on $\mathfrak{G}_1$ would be of order 2^n, which is impossible. Therefore $\mathrm{Ad}(F_2) \equiv 1$ on $\mathfrak{G}_1$.

4. THEOREM 4.1. *Let G be a compact connected Lie group, L a connected subgroup of maximal rank; then G/L possesses an invariant almost complex structure if and only if L is the connected component of the centralizer in G of a finite subgroup F, of its center, such that* $\mathrm{Ad}(F)$ *is a finite group of odd order.*

The sufficiency of this condition follows from Corollary 2.1. In this general case when L is not necessarily semi-simple, we proceed by proving the following two lemmas.

LEMMA 4.2. *Let G be compact, connected, L a connected subgroup of maximal rank such that G/L has an i.a.c.s.; if $L = T^a \times L_1$, local direct*

product, where T^a is toral and L_1 is connected semi-simple, then L is the centralizer in G of $T_a \cup F_1$, where F_1 is a finite subgroup of L, such that $\mathrm{Ad}(F_1)$ *is of odd order.*

PROOF. $L = T^a \times L_1$, $\mathfrak{L} = R^a \oplus \mathfrak{L}_1$, let $G_2 =$ centralizer of T^a in G, then G_2 is a connected subgroup of maximal rank in G, and

$$G_2 = T^a \times G_3, \qquad \mathfrak{G}_2 = R^a \oplus \mathfrak{G}_3, \qquad \text{where } G_3 \text{ is closed in } G_2.$$

By hypothesis there exists J, commuting with $\mathrm{Ad}(L)$ such that

$$J: \mathfrak{G}_2/\mathfrak{L} \to \mathfrak{G}_2/\mathfrak{L},$$

but $\mathfrak{L}_1 \subset \mathfrak{L}$, and there exists an isomorphism

$$\phi: \mathfrak{G}_2/\mathfrak{L} \to \mathfrak{G}_3/\mathfrak{L}_1.$$

Define J' on $\mathfrak{G}_3/\mathfrak{L}_1$ by $J' = \phi J \phi^{-1}$, then J' defines an almost complex structure on G_3/L_1. Then, by 3.5, L_1 is the connected component of the centralizer in G_3 of a finite subgroup F_1 of its center, such that $\mathrm{Ad}(F_1)$ is of odd order.

Now, if $x \in L$, then $x = tl_1$, where $t \in T^a$ and $l_1 \in L_1$. If $f_1 \in F_1$, then $(t \cdot l_1)f_1 = tf_1l_1 = f_1(tl_1)$ since t is central, and $f_1l_1 = l_1f_1$ since $l_1 \in L_1$. Thus x commutes with F_1, and x commutes with T^a.

Conversely, if x belongs to the connected component of the centralizer of $T^a \cup F_1$, then $x \in G_2$, so that $x = tg_3$. Moreover, since x commutes with F_1, we have $xf_1 = f_1x$, so that $tg_3f_1 = f_1tg_3 = tf_1g_3$, so that $g_3f_1 = f_1g_3$, thus $g_3 \in L_1$, and $x \in T^a \times L_1$, that is, $x \in L$.

LEMMA 4.3. *Let G be compact, connected Lie group and L a subgroup of G, such that L is the centralizer in G of a toral subgroup T^a. Then there exists an $x \in T^a$ such that* (1) *L is the connected component of the centralizer of x in G, and* (2) $\mathrm{Ad}(x)$ *is of odd order.*

PROOF OF (1). $T^a \subset T^b$, where T^b is a maximal torus of G. If θ_i, $i = 1, \cdots, m$, are the root forms, or angular parameters of T^b, and R^b is the covering space of T^b, then R^a, the covering space of T^a, is characterized by $R^a = \{x: \theta_i(x) = 0,\ i = 1, \cdots, m',\ m' \leqq m\}$, where θ_i, $i = 1, \cdots, m'$, are angular parameters of T^a. If $x \in T^a$, and $Z(x)$ denotes the connected component of the centralizer of x in G, then $L \subset Z(x)$. Moreover $Z(x)$ is minimal if $\theta_i(x) \neq 0$, $i = m'+1, \cdots, m$. But each plane $\theta_i(x) = 0$, $i = m'+1, \cdots, m$, determines a hyperplane of R^a, and the union of these finitely many hyperplanes cannot be all of R^a, thus there exist $x \in R^a$, $\theta_i(x) \not\equiv 0$, $i = m'+1, \cdots, m$.

PROOF OF (2). Given any Euclidean space R, the set of elements with coordinates whose denominators are all odd is dense in R, because the set of points with rational coordinates is dense in R. And

if $x=(r_1, \cdots, r_n)$ where $r_i=p_i/q_i$, p_i, q_i relatively prime, is a point with rational coordinates and

$$x_m = \left(\frac{2mp_1}{2mq_1+1}, \frac{2mp_2}{2mq_2+1}, \cdots, \frac{2mp_n}{2mq_n+1}\right),$$

then $\lim_{m\to\infty} x_m = r$.

Now let $\bar{\theta}_i$, $i=1, \cdots, b$, be a basis for R^b, consisting of root vectors. Thus if $x \in R^b$, then $\langle \bar{\theta}_i, x\rangle = \theta_i(x)$ defines an inner product on R^b, and if $x = \sum x_i\bar{\theta}_i$, $\langle \bar{\theta}_i, x\rangle = x_i$ so that all $\theta_i(x) \equiv 0 \bmod 1$, if x has integral coordinates in this basis, that is, $\mathrm{Ad}\, x = 1$.

Now if $x \in T^a$, such that $Z(x) = \mathfrak{L}$, then it does not lie on any of the planes $\theta_i(x) = 0$, $i = m'+1, \cdots$, which is a closed set, and there exists a point z with rational coordinates, with odd denominators, arbitrarily close to x, so that $Z(x) = \mathfrak{L}$. But $\mathrm{Ad}(z^k) = 1$, where $k =$ the least common multiple of the denominators of the coordinates of z. And, $\mathrm{Ad}(z^k) = \mathrm{Ad}^k(z)$.

Combining the last two lemmas, we have that since L is the centralizer of $x \in T^a$, such that $\mathrm{Ad}(x)$ is of odd order and L is the connected component of the centralizer of F_1, then L is the connected component of the centralizer, in G, of the group generated by x and F_1, whose adjoint is of odd order.

Bibliography

1. A. Borel and J. deSiebenthal, *Les sous-groupes fermés de rang maximum des groupes de Lie clos*, Comment. Math. Helv. **23** (1949), 200–221.

2. A. Frölicher, *Zur Differentialgeometrie der komplexen Strukturen*, Math. Ann. **129** (1955), 50–95.

3. R. Hermann, *Compact homogeneous almost complex spaces of positive characteristic*, Trans. Amer. Math. Soc. **83** (1956), 471–481.

4. H. C. Wang, *Closed manifolds with homogeneous complex structure*, Amer. J. Math. **76** (1954), 1–32.

Lake Forest College
and New York University