THE STRUCTURE OF HYPERREDUCIBLE TRIANGULAR ALGEBRAS

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Introduction. In [5] Kadison and Singer have defined triangular algebras of operators on a Hilbert space and have investigated a number of their properties with the major emphasis on classification and examples. It is the purpose of this paper to give a new construction for the hyperreducible algebras which gives some additional insight into their structure. In particular, Theorem 3 shows that any algebra $\mathfrak{T}$ of this form with diagonal $\mathfrak{A}$ can be written $\mathfrak{T}=\mathfrak{A}+\mathfrak{S}$, where $\mathfrak{S}$ is the weak closure of an increasing sequence of weakly closed nilpotent ideals.

1. **Notation and spectral theory.** $\mathfrak{A}$ will denote a fixed maximal abelian self-adjoint subalgebra of a factor $\mathfrak{B}$ acting on a separable Hilbert space $\mathfrak{H}$. For $X\in\mathfrak{B}$, define the operators L_X, R_X, D_X on $\mathfrak{B}$ by the equations $L_xY=XY$, $R_xY=YX$, and $D_xY=XY-YX$ for $Y\in\mathfrak{B}$. If the uniform topology is used on $\mathfrak{B}$, each of these operators is bounded and $\|L_x\|=\|R_x\|=\|X\|$. Furthermore, D_x is a derivation on $\mathfrak{B}$. Let $\mathfrak{L}$ and $\mathfrak{R}$ denote the sets of L_A, R_A, $A\in\mathfrak{A}$, respectively. Then $\mathfrak{L}$ and $\mathfrak{R}$ are commutative Banach algebras and each is isomorphic to $\mathfrak{A}$ under the natural mappings. Let $\mathfrak{C}$ be the uniformly closed algebra of operators on B generated by $\mathfrak{L}\cup\mathfrak{R}$. Thus $\mathfrak{C}$ contains the identity operator.

We let Δ denote the spectrum of $\mathfrak{A}$, i.e., the set of all homomorphisms of $\mathfrak{A}$ onto the complex numbers. Δ is a compact Hausdorff space under the Gel'fand topology and $\mathfrak{A}$, $\mathfrak{L}$, and $\mathfrak{R}$ all may be identified with the continuous complex-valued functions on Δ. If we let Γ denote the spectrum of $\mathfrak{C}$ then Γ is also compact and Hausdorff in the Gel'fand topology.

THEOREM 1. *Suppose* $\gamma\in\Gamma$ *and* α, β *are the restrictions of* γ *to* $\mathfrak{L}$ *and* $\mathfrak{R}$, *respectively. Then the mapping* $\gamma\to\gamma'=(\alpha,\beta)$ *is a homeomorphism of* Γ *onto the product space* $\Delta\times\Delta$.

For the proof we need two lemmas.

LEMMA 1. *For* X, $Y\in\mathfrak{B}$, $L_xR_y=0$ *implies* $X=0$ *or* $Y=0$.

PROOF. Let $\mathfrak{J}=\{z: L_zR_y=0\}$. Then $X\in\mathfrak{J}$ and $\mathfrak{J}$ is a weakly closed

Received by the editors January 25, 1963 and, in revised form, May 27, 1963.

two-sided ideal of $\mathfrak{B}$ so that either $\mathfrak{I}=\{0\}$ or $\mathfrak{I}=\mathfrak{B}$. In the first case $X=0$ and in the second $Y=0$.

LEMMA 2. *Let $\mathfrak{C}_0$ be the set of all finite linear combinations of the form $\sum\lambda_i L_{E_i}R_{F_i}$ where E_i and F_i are projections in $\mathfrak{A}$ and either $E_iE_j=0$ or $F_iF_j=0$ for $i\neq j$. Then $\mathfrak{C}_0$ is a dense subalgebra of $\mathfrak{C}$.*

PROOF. The usual operations with projections show fairly directly that $\mathfrak{C}_0$ is a subalgebra. In fact, with the obvious modifications, the proof is like that showing that the set of all finite linear combinations of characteristic functions of sub-rectangles (with sides parallel to the axes) of a given planar rectangle forms an algebra. From the spectral theorem applied to A and B it is easy to see that each L_AR_B $(A, B\in\mathfrak{A})$ can be approximated uniformly by elements of $\mathfrak{C}_0$ and thus $\mathfrak{C}_0$ is dense in $\mathfrak{C}$.

PROOF OF THEOREM 1. For $\gamma\in\Gamma$ and an element of $\mathfrak{C}_0$, $\gamma(\sum L_{E_i}R_{F_i}) = \sum\alpha(E_i)\beta(F_i)$. Since $\mathfrak{C}_0$ is uniformly dense in $\mathfrak{C}$ this means that γ is determined uniquely by α and β and thus the correspondence defined on Γ is one-one. If $\{\gamma_j\}$ is a directed sequence in Γ converging to γ and $\gamma_j=(\alpha_j, \beta_j)$, $\gamma'=(\alpha, \beta)$ then $\gamma_j(A)\rightarrow\gamma(A)$ for all $A\in\mathfrak{A}$ so that $\{\alpha_j\}$ and $\{\beta_j\}$ converge to α and β, respectively. Thus $\gamma_j'\rightarrow\gamma'$ and the mapping is continuous. Since Γ is compact, the image is compact and it only remains to prove that this image is all of $\Delta\times\Delta$.

Thus suppose $(\alpha, \beta)\in\Delta\times\Delta$ and $C=\sum\lambda_iL_{E_i}R_{F_i}\in\mathfrak{C}_0$. Let $\gamma(C) = \sum\lambda_i\alpha(E_i)\beta(F_i)$. If $\gamma(C)\neq 0$ there is a j with $\alpha(E_j)=\beta(F_j)=1$. For $i\neq j$ either $E_iE_j=0$ or $F_iF_j=0$ so that, in any case, $\alpha(E_i)\beta(F_i)=0$ and hence $\gamma(C)=\lambda_j$. Since $L_{E_j}R_{F_j}\neq 0$, there is an $X\in\mathfrak{B}$ with $\|X\|=1$ and $E_jXF_j=X$. Necessarily $E_iXF_i=0$ for $i\neq j$ and thus $C(X)=\lambda_jX$ so that $\|C\|\geqq|\lambda_j|=|\gamma(C)|$. The mapping $C\rightarrow\gamma(C)$ will be linear on $\mathfrak{C}_0$ and the argument above shows that it is well-defined and norm-decreasing. γ is clearly multiplicative on $\mathfrak{C}_0$ and thus can be extended uniquely to all of $\mathfrak{C}$ giving a homomorphism of $\mathfrak{C}$ also denoted by γ. Since $\gamma(L_1)=1$, γ is nontrivial and thus belongs to Γ. Since $\gamma'=(\alpha, \beta)$ the proof is complete.

2. **Construction of triangular algebras.**

DEFINITIONS. Because of the separability assumption there exists a self-adjoint element $A\in\mathfrak{A}$ such that every element of $\mathfrak{A}$ is a bounded measurable function of A. Thus $\mathfrak{A}=\{X: D_AX=0\}$. We choose a fixed A with these properties and, without any significant loss of generality, assume that the spectrum of A lies in the interval $[0, 1]$ and contains both end points. For $\gamma=(\alpha, \beta)\in\Gamma$ let $f(\gamma)=\gamma(D_A) =\alpha(A)-\beta(A)$. Thus, relative to $\mathfrak{C}$, the spectrum of D_A is given by the range of f, a subset of $[-1, 1]$. Since the range of f is real and compact, it does not separate the plane, and the general theory of Banach algebras implies that the spectrum of D_A, relative to the algebra of all bounded operators on $\mathcal{B}$, is also given by the range of f.

For real λ and $\epsilon>0$, let $S(\lambda, \epsilon)$ denote the linear manifold consisting of all $X\in\mathcal{B}$ for which there is a constant K_x such that $\|(D_A-\lambda)^n X\| \leqq \epsilon^n K_x$ for $n=1, 2, \cdots$. Then $S(\lambda, \epsilon)$ is invariant under all bounded operators on $\mathcal{B}$ which commute with D_A. For a compact subset M of the real numbers, let $S(M, \epsilon)$ be the closure in the weak operator topology of $\mathcal{B}$ of the manifold spanned by $\{S(\lambda, \epsilon):\lambda\in M\}$ and let $S(M)=\bigcap_{\epsilon>0} S(M, \epsilon)$. Finally, for any Borel set N of the real numbers, let $S(N)$ be the weak closure of the manifold spanned by $\{S(M): M \subseteq N, M \text{ compact}\}$. Thus $S(N)$ is invariant under all bounded operators on $\mathcal{B}$ commuting with D_A. In particular, $\mathcal{A}\subseteq S(\{0\})$.

LEMMA 3. *Suppose M is a real Borel set and $|\lambda|>1$ for all $\lambda\in M$. Then $S(M)=\{0\}$.*

PROOF. The proof reduces to the case when M is compact. The hypothesis implies M lies within the resolvent set for D_A. Thus for $\gamma\in M$, $D_A-\lambda I$ has a bounded inverse $B(\lambda)$ and the mapping $\lambda\to B(\lambda)$ is holomorphic on an open set containing M. This implies the existence of a constant K such that $\|B(\lambda)\|\leqq K$ for all $\lambda\in M$. Choose ϵ with $0<\epsilon<K^{-1}$. For $\lambda\in M$ and $X\in S(\lambda, \epsilon)$, $\|X\|=\|B(\lambda)^n(D_A-\lambda)^n X\| \leqq (K\epsilon)^n K_x$ for all $n>0$ and thus $\|X\|=0$. Thus $S(M, \epsilon)=\{0\}$ and the same is true for $S(M)$.

LEMMA 4. *For Borel sets M, N; $S(M)S(N)\subseteq S(M+N)$ and $S(M)^* =S(-M)$.*

PROOF. Suppose $\epsilon>0$ and $X\in S(\lambda, \epsilon)$, $Y\in S(\mu, \epsilon)$. Since D_A is a derivation,

$$\|(D_A-(\lambda+\mu))^n XY\| = \left\| \sum_n (C_m((D_A-\lambda)^m X)(D_A-\mu)^{n-m}Y)\right\|$$
$$\leqq \sum_n C_m \epsilon^m K_X \epsilon^{n-m} K_Y = (2\epsilon)^n K_X K_Y.$$

Thus $S(\lambda, \epsilon)S(\mu, \epsilon)\subseteq S(\lambda+\mu, 2\epsilon)$.

Suppose now that M, N are compact, $\lambda_i\in M$, $\mu_j\in N$, $X_i\in S(\lambda_i, \epsilon)$, $Y_j\in S(\mu_j, \epsilon)$, and $X=\sum X_i$, $Y=\sum Y_j$. Then $XY=\sum\sum X_i Y_j$ so that $XY\in S(M+N, 2\epsilon)$. The sets of X and Y obtained in this way are weakly dense in $S(M, \epsilon)$ and $S(N, \epsilon)$, respectively. Since, in the weak operator topology, multiplication is continuous in each factor separately, we then have $S(M)S(N)\subseteq S(M, \epsilon)S(N, \epsilon)\subseteq S(M+N, 2\epsilon)$.

Since ϵ was arbitrary, this implies $S(M)S(N)\subseteq S(M+N)$ for compact M and N. The general case follows immediately from this.

Induction on n shows that $((D_A-\lambda)^nX)^*=(-1)^n((D_A+\lambda)^nX^*)$ for real λ and $X\in\mathfrak{B}$. Thus $S(\lambda, \epsilon)^*=S(-\lambda, \epsilon)$. A proof like that used above shows that $S(M)^*=S(-M)$.

Corollary. *For each* $\lambda>0$, $S([\lambda, 1])$ *is a weakly closed nilpotent subalgebra of* $\mathfrak{B}$. *In fact, if n is any integer with* $n\lambda>1$, *the product of any n factors taken from* $S([\lambda, 1])$ *is zero.*

Definitions. Let $\mathcal{S}_0=\bigcup_{\lambda>0} S([\lambda, 1])$ and $\mathcal{S}=S((0, 1])$. Let $\mathfrak{I}$ be the algebraic sum of $\mathfrak{A}$ and $\mathcal{S}$. By virtue of Lemma 4 the following assertions are evident:

(1) $\mathfrak{I}$ is a subalgebra of $\mathfrak{B}$.

(2) $\mathcal{S}_0$ and $\mathcal{S}$ are two-sided ideals of $\mathfrak{I}$.

(3) Every element of $\mathcal{S}_0$ is nilpotent.

(4) $\mathcal{S}$ is the weak closure of $\mathcal{S}_0$.

It will be shown below that $\mathfrak{I}$ is a maximal hyperreducible triangular algebra with diagonal $\mathfrak{A}$.

Let E be the real spectral measure associated with A and let $E_\lambda=E((-\infty, \lambda))$. Then $E_\lambda=\sup E_\mu$ where $\mu<\lambda$ so that $E_0=0$. Also $E_\lambda=1$ for $\lambda>1$ and the set of E_λ generates $\mathfrak{A}$ as a von Neumann algebra.

For λ real and $\epsilon>0$ let $\mathfrak{F}(\lambda, \epsilon)$ be the set of vectors $u\in\mathfrak{H}$ such that $\|(A-\lambda)^nu\|\leqq\epsilon^nK_u$ for some constant K_u and $n=1, 2, \cdots$. If we let $\mathfrak{F}(N, \epsilon)$ be the closed subspace spanned by $\{\mathfrak{F}(\lambda, \epsilon):\lambda\in N\}$ then it follows from a result in [3, pp. 66–69] that if N is compact, the range of $E(N)$ equals $\mathfrak{F}(N)=\bigcap_{\epsilon>0}\mathfrak{F}(N, \epsilon)$. For a Borel set N we use $\mathcal{E}(N)$ to denote the range of the projection $E(N)$.

Theorem 2. *For Borel sets M and N*, $S(M)\mathcal{E}(N)\subseteq\mathcal{E}(M+N)$.

Proof. Suppose λ, μ are real and $\epsilon>0$. Choose $u\in\mathfrak{F}(\mu, \epsilon)$ and $X\in S(\lambda, \epsilon)$. Then

$$\begin{aligned}\|(A-(\lambda+\mu))^nXu\| &= \|((D_A-\lambda)+(R_A-\mu))^nXu\| \\ &= \left\|\sum_n C_m(D_A-\lambda)^{n-m}X(A-\mu)^mu\right\| \\ &\leqq \sum_n C_m\epsilon^mK_X\epsilon^{n-m}K_u = (2\epsilon)^nK_XK_u.\end{aligned}$$

Thus $S(\lambda, \epsilon)\mathfrak{F}(\mu, \epsilon)\subseteq\mathfrak{F}(\lambda+\mu, 2\epsilon)$.

An argument like that used in the proof of Lemma 4 can now be used to prove the assertion of the theorem when M and N are compact. By using approximations with compact subsets the general assertion follows from this.

COROLLARY. *For any real* λ *the range of* $1-E_\lambda$ *is invariant under* $\mathfrak{J}$.

LEMMA 5. (1) *Suppose* $0\leqq\mu<\lambda\leqq 1$. *Then* $(1-E_\lambda)\mathfrak{B}E_\mu\subseteq\mathfrak{S}_0$.
(2) *For any* λ, $(1-E_\lambda)\mathfrak{B}E_\lambda\subseteq\mathfrak{S}$.

PROOF. (1) Suppose $\epsilon>0$. Let $\lambda_0=\lambda$ and choose $\lambda_1, \cdots, \lambda_n$ with $\lambda_n>1$ such that each of the intervals $[\lambda_0, \lambda_1), \cdots, [\lambda_{n-1}, \lambda_n)$ has length less than ϵ. Let $E_i=E([\lambda_{i-1}, \lambda_i))$, $i=1, \cdots, n$. Then $1-E_\lambda=\sum E_i$. Similarly we can partition $[0, \mu)$ into m disjoint intervals of length less than ϵ and express E_μ as the sum of m mutually orthogonal projections $F_j\in\mathfrak{A}$. Choose arbitrary points α_i, β_j in the ith and jth intervals of the partitions of $[\lambda_0, \lambda_n)$ and $[0, \mu)$, respectively. Then $\alpha_i-\beta_j\geqq\lambda-\mu>0$.

For $X\in\mathfrak{B}$, $(1-E_\lambda)XE_\mu=\sum\sum E_iXF_j$. For a positive integer k,

$$\begin{aligned}\|(D_A-(\alpha_i-\beta_j))^kE_iXF_j\| &= \|((L_A-\alpha_i)-(R_A-\beta_j))^kE_iXF_j\|\\ &\leqq \sum_k C_p\|(A-\alpha_i)^pE_i\|\,\|X\|\,\|(A-\beta_j)^{k-p}F_j\|\\ &\leqq (2\epsilon)^k\|X\|.\end{aligned}$$

Thus $E_iXF_j\in S(\alpha_i-\beta_j, 2\epsilon)\subseteq S([\lambda-\mu, 1])\subseteq\mathfrak{S}_0$.

(2) For any X and $0\leqq\lambda\leqq 1$, $(1-E_\lambda)XE_\lambda$ is a weak limit of operators of the form $(1-E_\lambda)XE_\mu$ where $\mu<\lambda$, hence lies in $\mathfrak{S}$.

THEOREM 3. $\mathfrak{J}$ *is a maximal hyperreducible triangular algebra with diagonal* $\mathfrak{A}$. *Moreover, every algebra of this type (acting on a separable space) is obtained by a construction like that given above.*

PROOF. Let $\mathfrak{J}_1$ be the set of all $X\in\mathfrak{B}$ which leave the range of $1-E_\lambda$ invariant for all λ. It is shown in [**5**, Theorem 3.1.1] that $\mathfrak{J}_1$ is a maximal hyperreducible triangular algebra and we have proved that $\mathfrak{J}\subseteq\mathfrak{J}_1$. It remains to prove $\mathfrak{J}_1\subseteq\mathfrak{J}$. For the proof we use the diagonalization process developed by von Neumann and generalized by Kadison and Singer in [**4**, pp. 386–387].

Let $\{\lambda_n\}$ be a countable dense subset of $[0, 1]$ and $E_n=E_{\lambda_n}$. Then $\{E_n\}\cup\{1\}$ generates $\mathfrak{A}$ as a von Neumann algebra. Define the projection P_n on $\mathfrak{B}$ by $P_n(X)=E_nXE_n+(1-E_n)X(1-E_n)$. Then $\|P_n(X)\|\leqq\|X\|$ and P_n leaves $\mathfrak{J}_1$ invariant. For $X\in\mathfrak{J}_1$, $X=P_n(X)+(1-E_n)XE_n$ and the latter term is in $\mathfrak{S}$ by virtue of Lemma 5. Let $X_0=X$ and $X_n=P_n(X_{n-1})$ for $n\geqq 1$. Then $X_n\in\mathfrak{J}_1$ and, by induction on n, we have $X=X_n+S_n$ where $S_n\in\mathfrak{S}$. Because of the compactness of the unit sphere of $\mathfrak{B}$ in the weak topology a subsequence $\{X_{n_k}\}$

converges weakly to an element $B\in\mathfrak{B}$ and thus $\{S_{n_k}\}$ converges weakly to some $S\in\mathfrak{S}$. Since B will commute with all E_n, $B\in\mathfrak{A}$ and hence $X=B+S$ lies in $\mathfrak{T}$ so that $\mathfrak{T}_1\subseteq\mathfrak{T}$.

If $\mathfrak{T}$ is any maximal hyperreducible triangular algebra with diagonal $\mathfrak{A}$, a modification of the proof of Theorem 3.3.1 in [5] shows that it is possible to construct a spectral family $\{E_\lambda: 0\leqq\lambda\leqq 1\}$ of projections in $\mathfrak{A}$ such that $\mathfrak{T}$ is the set of all $X\in\mathfrak{B}$ with $E_\lambda X=E_\lambda XE_\lambda$ for all λ. If we define A by $A=\int\lambda dE_\lambda$, then $\mathfrak{A}=\{X: D_A X=0\}$, and if we use this A in the construction, the result will be $\mathfrak{T}$, as shown in the preceding part of the proof.

3. Determination of $\mathfrak{A}\cap\mathfrak{S}$.

REMARK. In connection with the structure of $\mathfrak{T}$ the question arises as to necessary conditions in order that the sum $\mathfrak{A}+\mathfrak{S}$ be direct. A complete solution is given here for the case when $\mathfrak{B}$ is of Type I or II. A question which seems to be related to this arises regarding the possibility of expressing each $X\in\mathfrak{T}$ as $X=B+S$ where $B\in\mathfrak{A}$ and S is quasi-nilpotent. The answer to this is not known to us and, because of the nonadditivity of quasi-nilpotence, it is conceivable that it can be accomplished even when $\mathfrak{A}\subseteq\mathfrak{S}$.

LEMMA 6. *Suppose $\mathfrak{B}$ is the algebra of all bounded operators on $\mathfrak{H}$ and the point spectrum of A is empty. Then $\mathfrak{A}\subseteq\mathfrak{S}$.*

PROOF. Let $\mathfrak{g}=\mathfrak{A}\cap\mathfrak{S}$. Then $\mathfrak{g}$ is an ideal of $\mathfrak{A}$ and thus it is sufficient to prove that $1\in\mathfrak{S}$. Because of [5, Theorem 3.3.1] we may assume that $\mathfrak{A}$ is the algebra of all bounded measurable functions in $L^2([0, 1])$ and E_λ is multiplication by the characteristic function of the interval $[0, \lambda)$.

Let $\mathcal{E}$ be the set of all operators on L^2 which are finite sums $\sum f_i\otimes g_i$ where f_i, $g_i\in L^2$ and $(f_i\otimes g_i)h=(h, g_i)f_i$. By making use of the canonical trace on $\mathfrak{B}$, $\mathcal{E}$ can be identified with the set of all weakly continuous linear functionals on $\mathfrak{B}$ [1, p. 388]. Let $\mathcal{P}=\{T: T\in\mathcal{E}, \operatorname{Tr}(TX)=0$ for all $X\in\mathfrak{S}\}$. Since $\mathfrak{S}$ is weakly closed, $\mathfrak{S}=\{X: \operatorname{Tr}(TX)=0$ for all $T\in\mathcal{P}\}$. Thus $1\in\mathfrak{S}$ if and only if $\operatorname{Tr}(T)=0$ for all $T\in\mathcal{P}$.

Suppose $T\in\mathcal{P}$. For f, $g\in L^2$, Lemma 5 implies $(1-E_\lambda)f\otimes gE_\lambda\in\mathfrak{S}$ so that $\operatorname{Tr}(T(1-E_\lambda)f\otimes gE_\lambda)=0$ and this implies $(E_\lambda T(1-E_\lambda)f, g)=0$. Since f, g were arbitrary this shows that $E_\lambda T(1-E_\lambda)=0$ for any λ and thus $T\in\mathfrak{T}$. Since T is of trace class there is a function K in L^2 of the unit square such that $Tf(x)=\int K(x, y)f(y)\,dy$ for $f\in L^2$. Since $T\in\mathfrak{T}$ it is easy to see that K is of Volterra type, i.e., $K(x, y)=0$ for $y>x$. A well-known theorem of integral equations [6, pp. 10–11] implies T is quasi-nilpotent and, since the range of T is finite-dimensional, this means T is nilpotent so that $\operatorname{Tr}(T)=0$.

THEOREM 4. (1) *If* $\mathcal{B}$ *is a factor of Type* II, $\mathcal{A}\cap\mathcal{S}=\{0\}$.

(2) *Suppose* $\mathcal{B}$ *is of Type* I_∞. *Let* M *be the set of characteristic values of* A *and* $E=E([0,1]-M)$. *Then* $\mathcal{A}\cap\mathcal{S}=E\mathcal{A}$. *Thus* $\mathcal{A}\cap\mathcal{S}=\{0\}$ *if and only if* A *has pure point spectrum.*

PROOF. (1) We consider only the case when $\mathcal{B}$ is finite. The infinite case can be deduced from this or obtained by a refinement of the argument given here. For $X, Y\in\mathcal{B}$ let $(X, Y)=\mathrm{Tr}(XY^*)$ where Tr is the canonical trace on $\mathcal{B}$. Then $\mathcal{B}$ becomes a pre-Hilbert space. Since $\mathcal{A}\subseteq S(\{0\})$ the assertion follows from a more general result: If M and N are disjoint Borel sets then $(S(M), S(N))=0$. By using approximations with compact subsets we may assume both M and N are compact. Then $M-N$ is a compact set bounded away from zero and $S(M)S(N)^*\subseteq S(M-N)$. Since $S(M-N)$ is spanned by nilpotent elements of $\mathcal{B}$ it is sufficient to prove that any nilpotent $X\in\mathcal{B}$ has trace zero. However, this is a consequence of a result proved in [**2**, p. 108] where it is shown that $\mathrm{Tr}(X)$ belongs to the convex hull of the spectrum of X.

(2) Let $\mathcal{J}=\mathcal{S}\cap\mathcal{A}$. Then $\mathcal{J}$ is a weakly closed ideal of $\mathcal{A}$. Suppose μ is a characteristic value for A and $P=E(\{\mu\})$. Then P is a minimal projection in $\mathcal{A}$ and thus either $P\in\mathcal{J}$ or $P\mathcal{J}=0$. Choose $\lambda>0$ and $S\in S([\lambda, 1])$. Then Theorem 2 implies $SP(\mathcal{H})\subseteq E(\{\mu\}+[\lambda, 1])(\mathcal{H})$ so that $Pv=v$ implies $(Sv, Pv)=0$. But then $\|(P-S)v\|^2=\|v\|^2+\|Sv\|^2 \geq\|v\|^2$. Since λ was arbitrary this shows that P is not contained in the strong closure of $\mathcal{S}_0$. However, the result of [1, Note 1] shows that the strong and weak closures coincide and thus $P\notin\mathcal{S}$. Then $P\mathcal{J}=0$ and it follows that $\mathcal{J}\subseteq E\mathcal{A}$. Let $\mathfrak{J}'=E\mathfrak{J}E$. Then $\mathfrak{J}'$ is a maximal hyper-reducible triangular algebra of operators with diagonal $E\mathcal{A}$ on the Hilbert space $E\mathcal{H}$. Since EA has no point spectrum on $E\mathcal{H}$, Lemma 6 implies $E\mathcal{A}\subseteq E\mathcal{S}E$ and hence $E\mathcal{A}\subseteq\mathcal{J}$.

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