THE ANNIHILATOR OF A KNOT MODULE

R. H. CROWELL

1. Introduction. Let \( Z_t \) be the integral group ring of an infinite cyclic multiplicative group generated by \( t \). If \( K \subseteq S^3 \) is a tame knot with group \( G = \pi_1(S^3 - K) \) and commutator subgroup \( G' \), then \( G/G' \) is infinite cyclic, and its integral group ring is therefore isomorphic to \( Z_t \). An important set of invariants of \( K \), discussed in [3], are the knot polynomials \( \Delta_1, \Delta_2, \ldots \), which belong to \( Z_t \) and satisfy \( \Delta_{k+1} | \Delta_k \) and \( |\Delta_k(1)| = 1 \). The abelianized commutator subgroup \( G'/G'' \), if written additively, is a \( Z_t \)-module: Let \( u: G \rightarrow G/G' \) and \( \mu: G' \rightarrow G'/G'' \) be the abelianizing maps, and \( t = vg \) a generator of \( G/G' \). The action of \( t \) on an arbitrary \( \mu k \in G'/G'' \) is defined by \( t \cdot \mu k = \mu(kg^{-1}) \). If a column is deleted from the Alexander matrix of any Wirtinger presentation of \( G \), it follows directly from [4, §7] that the resulting square matrix is a relation matrix for \( G'/G'' \). In this paper we show that the annihilating ideal of the \( Z_t \)-module \( G'/G'' \) is principal and generated by \( \Delta_1/\Delta_2 \).

2. Noetherian modules. Let \( R \) be a commutative noetherian ring, and \( A \) a finitely generated \( R \)-module. The set of all \( a \in R \) such that \( a \cdot a = 0 \) for all \( a \in A \) is an ideal, which we call the annihilator of \( A \) and denote by \( \mathfrak{a} \). Since any finitely generated module over \( R \) is noetherian [2, p. 15], there exists an exact sequence

\[
0 \rightarrow X_1 \rightarrow X_0 \rightarrow X \rightarrow 0
\]

of \( R \)-morphisms, called a presentation sequence, in which \( X_0 \) and \( X_1 \) are free \( R \)-modules with bases \( (x_1, \ldots, x_n) \) and \( (y_1, \ldots, y_m) \), respectively. The matrix \( (\alpha_{ij}) \) defined by

\[
d_y = \sum_{j=1}^n \alpha_{ij} \cdot x_j, \quad i = 1, \ldots, m,
\]

is a relation matrix for \( A \). For each positive integer \( k \), the \( k \)th elementary ideal\(^2\) of \( (\alpha_{ij}) \) is the ideal \( \mathfrak{e}_k \) generated by the determinants of all \( (n-k+1) \times (n-k+1) \) submatrices of \( (\alpha_{ij}) \). We adopt the convention that \( \mathfrak{e}_k = 0 \) if \( m < (n-k+1) \), and \( \mathfrak{e}_k = R \) if \( (n-k+1) < 1 \). Obviously,

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\(^3\) We have numbered the elementary ideals so that \( \mathfrak{e}_k \) corresponds to \( \Delta_k \) in the application to knot modules. This numbering differs from both [3] and [7].
It is proved in [7, p. 90] that the elementary ideals depend only on $A$. Moreover,

(2.1) $e_t \subseteq a$,
(2.2) $\sqrt{e_t} = \sqrt{a}$.

A proof of (2.1), is given in [7, p. 93]. To establish (2.2), we first prove that if $R$ is an integral domain and $e_t = 0$, then $a = 0$. Let $K$ be the quotient field of $R$, and apply $K \otimes_R$ to a presentation sequence (1). Then $1 \otimes d$ is a linear transformation of finite dimensional vector spaces, and, because $e_t = 0$, its rank is $\leq n - 1$. Hence $K \otimes_R A \neq 0$, and it therefore contains an element $k \otimes_R a \neq 0$. It follows that zero is the only element in $R$ that can annihilate $a$, and so $a = 0$.

To prove (2.2), it suffices to show that every prime ideal $p$ that contains $e_t$ also contains $a$. Consider $\phi: R \rightarrow R/p$, and apply $R/p \otimes_R$ to a presentation sequence (1). In the resulting sequence the matrix of $1 \otimes d$ is $(\phi \alpha_i \epsilon_i)$ and is a relation matrix for $R/p \otimes_R A$. Its 1st elementary ideal is $\phi \epsilon_t = 0$. Since $R/p$ is an integral domain, the above lemma implies that the annihilator of $R/p \otimes_R A$, which contains $a$, is zero. Hence $a \subseteq p$, and the proof is complete.

We define the deficiency of $A$ to be the maximum of all numbers $n - m$ such that there exists an $m \times n$ relation matrix for $A$. Notice that the deficiency of $A$ cannot exceed the number of trivial elementary ideals of $A$.

(2.3) Suppose that $R$ is an integral domain and that $A$ has nonnegative deficiency. If $\pi \cdot a = 0$ for some prime $\pi \in R$ and nonzero $a \in A$, then $e_t \subseteq (\pi)$.

Proof. Since any number of zero rows can be adjoined to a relation matrix, we may assume that $A$ has a presentation sequence (1) for which $m = n$. Choose $\alpha_1, \cdots, \alpha_n \in R$ so that $e(\sum_{j=1}^{n} \alpha_j x_j) = a$. Since $\pi \cdot a = 0$,

$$
\sum_{i=1}^{n} \pi \alpha_j \cdot x_j = d \sum_{i=1}^{n} \beta_i \cdot y_i = \sum_{j=1}^{n} \left( \sum_{i=1}^{n} \beta_i \alpha_{ij} \right) \cdot x_j.
$$

Hence

$$
\pi \alpha_j = \sum_{i=1}^{n} \beta_i \alpha_{ij}, \quad j = 1, \cdots, n.
$$

Writing $\phi: R \rightarrow R/(\pi)$, we obtain

$$
\phi(\pi \alpha_j) = 0 = \sum_{i=1}^{n} \phi \beta_i \phi \alpha_{ij}, \quad j = 1, \cdots, n.
$$
It follows that either $\phi \beta_1 = \cdots = \phi \beta_n = 0$ or $\det(\phi \alpha_{ij}) = \phi \det(\alpha_{ij}) = 0$. Since $R$ is an integral domain, the first alternative implies $a = 0$ and is therefore false. We conclude that $\phi \epsilon_1 = \phi(\det(\alpha_{ij})) = (\phi \det(\alpha_{ij})) = 0$. Hence $\epsilon_1 \subseteq (\pi)$, and the proof is complete.

The preceding result is false without the requirement that $A$ have nonnegative deficiency. Consider the exact sequence $0 \rightarrow (2, t) \rightarrow \mathbb{Z}[t] \rightarrow \mathbb{Z}_2 \rightarrow 0$, and take the polynomial ring $\mathbb{Z}[t]$ for $R$, $\mathbb{Z}_2$ for $A$, $2$ for $\pi$, and $1 \in \mathbb{Z}_2$ for $a$. Then $\pi \cdot a = 0$ and $\epsilon_1 = a = (2, t)$, which is not contained in $(\pi)$.

3. Knot modules. Let $A$ be a finitely generated $\mathbb{Z}t$-module. Since $\mathbb{Z}t$, besides being noetherian, is a unique factorization domain, any ideal $b$ is contained in a smallest principal ideal $b^*$, equal to the intersection of all principal ideals containing $b$, see [3, Chapter VIII, §2]. The $k$th knot polynomial $\Delta_k$ is the generator, determined to within a unit multiple, of $\epsilon_k^*$. Clearly $\Delta_{k+1} \mid \Delta_k$. A nonzero element $\alpha = \sum n_k t^k$ in $\mathbb{Z}t$ is primitive if its coefficients \{n_k\} are relatively prime.

Let $Q$ be the field of rational numbers. Then $Q \mathbb{Z}t \cong Q \otimes \mathbb{Z}t \mathbb{Z}t$, the group ring with rational coefficients, is a principal ideal domain which contains $\mathbb{Z}t$.

(3.1) Let $b$ be an ideal in $\mathbb{Z}t$. If $\beta$ generates $b^*$, then $\beta$ also generates the smallest ideal in $Q \mathbb{Z}t$ that contains $b$.

Proof. We may assume $b \neq 0$. Clearly $b \subseteq b^* \subseteq (\beta)_{Q \mathbb{Z}t}$. Conversely, let $c$ be any ideal in $Q \mathbb{Z}t$ containing $b$, and choose a generator $\gamma$ which is in $\mathbb{Z}t$ and is primitive. For any $\alpha \in b$, we have $\alpha = \gamma \delta$ for $\delta \in Q \mathbb{Z}t$. Gauss’ lemma [1, p. 79] that the product of two primitive elements is primitive implies that in fact $\delta \in \mathbb{Z}t$, and so $b \subseteq (\gamma)_{\mathbb{Z}t}$. Since $b^*$ is minimal, $b^* = (\beta)_{\mathbb{Z}t} \subset (\gamma)_{\mathbb{Z}t}$. This implies $(\beta)_{Q \mathbb{Z}t} \subset (\gamma)_{Q \mathbb{Z}t} = c$ and completes the proof.

Consider a presentation sequence (1) for $A$ with relation matrix $(\alpha_{ij})$. Applying $Q \mathbb{Z}t$, we obtain a presentation sequence for $Q \otimes \mathbb{Z}t A$ with the same relation matrix except that the entries $\alpha_{ij}$ are now considered to be in $Q \mathbb{Z}t$. It follows by (3.1) that $\Delta_k$ generates the $k$th elementary ideal of the $Q \mathbb{Z}t$-module $Q \otimes \mathbb{Z}t A$. Hence the structure theorem for finitely generated modules over a principal ideal domain implies that:

(3.2) The annihilator of $Q \otimes \mathbb{Z}t A$ is generated by $\Delta_1/\Delta_2$.

Since $A \cong \mathbb{Z}t \otimes \mathbb{Z}t A$ under $a \rightarrow 1 \otimes \mathbb{Z}t a$, we obtain

$$Q \otimes \mathbb{Z}t A \cong Q \otimes (\mathbb{Z}t \otimes \mathbb{Z}t A) \cong (Q \otimes \mathbb{Z}t) \otimes \mathbb{Z}t A \cong Q \otimes \mathbb{Z}t A$$

under the map $g$ defined by $g(q \otimes \mathbb{Z}t a) = q \otimes \mathbb{Z}t a$. We therefore get the consistent diagram
where \(fa = 1 \otimes_Z a\) and \(ha = 1 \otimes_{\mathbb{Z}_t} a\).

(3.3) Let \(A\) be torsion free as a \(\mathbb{Z}\)-module. Then \(\Delta_i\) is primitive if it is nonzero, and the annihilator \(a \subseteq \mathbb{Z}_t\) is the principal ideal generated by \(\Delta_i/\Delta_2\).

Proof. If \(\Delta_i = 0\), then \(e_i = 0\) and by (2.2) also \(a = 0\). So we assume \(\Delta_i \neq 0\), and consequently \(e_i\) and \(a\) are also nonzero. If \(\Delta_i\) is not primitive, it is divisible by some prime integer \(p\). Hence \(e_i \subseteq (\Delta_i) \subseteq (p)\), and the latter is a prime ideal (Gauss' lemma). Hence \(a \subseteq (p)\) by (2.2).

Choose \(\alpha \neq 0\) in \(a\) and write \(\alpha = p^i\beta\), where \(p \nmid \beta\). Then \(\beta \in a\), so there exists \(\beta \in A\) with \(\beta \cdot a \neq 0\). Some one of the elements \(p^i\beta \cdot a, j = 0, \ldots, k - 1\), is nonzero and of order \(p\). This proves the first assertion.

Set \(\delta = \Delta_i/\Delta_2\). Since \(A\) is a torsion free abelian group, the map \(f\) in the diagram (2), and consequently \(h\) as well, is a monomorphism [2, p. 130]. For any \(a \in A\), we have \(0 = \delta \cdot ha = \delta \cdot (1 \otimes_{\mathbb{Z}_t} a) = 1 \otimes_{\mathbb{Z}_t} \delta \cdot a = h(\delta \cdot a)\), and so \(\delta \cdot a = 0\). Hence \(\delta \subseteq a\). Conversely, suppose that \(\alpha \in a\). Then for any \(\beta \in \mathbb{Z}_t\) and \(a \in A\), we have \(\alpha \cdot (\beta \otimes_{\mathbb{Z}_t} a) = (\beta \otimes_{\mathbb{Z}_t} \alpha \cdot a) = 0\), and so by (3.2) we have \(\alpha = \gamma \delta\) for some \(\gamma \in \mathbb{Z}_t\). But \(\delta\) is primitive because \(\Delta_i\) is, so Gauss' lemma again implies \(\gamma \in \mathbb{Z}_t\). We conclude that \(a = (\delta)_{\mathbb{Z}_t}\), and the proof is complete.

A partial converse to (3.3) is:

(3.4) If \(A\) has nonnegative deficiency and \(\Delta_i \neq 0\) is primitive, then \(A\) is a torsion free abelian group.\(^3\)

Proof. In fact, the deficiency must be zero, and \(e_i = (\Delta_i)\). Suppose \(A\) contains a nonzero element of prime order \(p\). As we have noted above, \(p\) generates a prime ideal of \(\mathbb{Z}_t\). It follows by (2.3) that \((\Delta_i) = e_i \subseteq (p)\), and so \(\Delta_i\) is not primitive.

The knot module \(G'/G''\) in the introduction has deficiency zero, and its 1st polynomial is nonzero and primitive because \(|\Delta_i(1)| = 1\). It follows from (3.4) that \(G'/G''\) is torsion free as a \(\mathbb{Z}\)-module and from (3.3) that, as a \(\mathbb{Z}_t\)-module, its annihilator is \((\Delta_i/\Delta_2)\).

The square knot, for example, has polynomials \(\Delta_1 = (t^2 - t + 1)^2\) and \(\Delta_2 = t^2 - t + 1\); so the annihilator \(a\) is the prime ideal generated by \(t^2 - t + 1\). For the knot \(8_{21}\) [6, pp. 41, 71], however, \(\Delta_1 = (t^2 - t + 1)^2\) and \(\Delta_2 = 1\), and \(a\) is therefore the primary ideal generated by \((t^2 - t + 1)^2\).

\(^3\) This is also proved in [5].
Bibliography


Dartmouth College

NOTE ON POINTWISE PERIODIC SEMIGROUPS

ANNE L. HUDSON

An element $x$ in a semigroup $S$ is said to be periodic if there exists a positive integer $n$ such that $x^{n+1} = x$, and the least such $n$, $p(x)$, is the period of $x$. $S$ is pointwise periodic if each $x$ in $S$ is periodic. In [4], A. D. Wallace asks the following question concerning pointwise periodic topological semigroups.

**Problem 3:** If $S$ is a pointwise periodic semigroup and is topologically an $n$-cell, is it possible that $S \setminus E$ is nonempty and $p(x) > 2$ and constant on $S \setminus E$?

It will be shown that in a slightly more general situation than that of the above problem, it necessarily follows that $p(x) = 2$ on $S \setminus E$.

The following notation will be used throughout this paper. For a semigroup $S$, $E = \{x : x \in S, x^2 = x\}$ and for $e \in E$, $H(e)$ is the maximal subgroup of $S$ containing the idempotent $e$. $H = \bigcup \{H(e) : e \in E\}$ and functions $\gamma$ and $\theta$ are defined as in [5], that is, for $x \in H$, $\gamma(x)$ is the idempotent of the unique maximal subgroup to which $x$ belongs and $\theta(x)$ is the inverse of $x$ in this group.

The following theorem will be proved:

**Theorem.** Let $S$ be a compact semigroup with the properties:

1. $S = H$,
2. for $e \in E$, $H(e)$ is totally disconnected,

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