ON THE NUMBER OF IRREDUCIBLE MODULAR REPRESENTATIONS OF A FINITE GROUP

IRVING REINER[1]

1. Let K be a field of characteristic p, and let G be a finite group. Denote by n the L.C.M. of the orders of the p-regular elements of G, and let δ be a primitive nth root of 1 over K. Each K-automorphism of $K(\delta)$ is determined by a map $\delta \to \delta^t$ for some integer t, taken modulo n. The multiplicative group T of all such exponents $t \pmod{n}$ is isomorphic to the Galois group of $K(\delta)$ over K.

Two p-regular elements $a, b \in G$ are called *K-conjugate* if $b^t = x^{-1}ax$ for some $x \in G$ and some $t \in T$. This defines an equivalence relation, relative to which the p-regular elements of G are partitioned into p-regular K-conjugacy classes.

The following is due to Berman [**2**].

THEOREM. *The number of irreducible K-representations of G equals the number of p-regular K-conjugacy classes of G.*

The purpose of this note is to present a simplified proof of Berman's theorem by making systematic use of Brauer characters. Let ω denote a primitive nth root of 1 over the rational field Q. To each K-representation F of G corresponds a Brauer character ϕ, defined on the p-regular elements of G, as follows: For p-regular $a \in G$, let the characteristic roots of $F(a)$ be $\delta^{m_1}, \cdots, \delta^{m_q}$, and set

$$\phi(a) = \omega^{m_1} + \cdots + \omega^{m_q}.$$

We refer the reader to [**4**] for the basic properties of Brauer characters, as well as for the other results used below.

Now let $\lambda_1, \cdots, \lambda_k$ be the Brauer characters of a full set of irreducible representations of G in the algebraic closure of K. Then $\lambda_1, \cdots, \lambda_k$ are linearly independent over $Q(\omega)$, that is, if $\alpha_1, \cdots, \alpha_k \in Q(\omega)$ are such that $\sum \alpha_i \lambda_i(g) = 0$ for all p-regular $g \in G$, then each $\alpha_i = 0$.

Finally, let U be a K-representation of a subgroup H of G, with Brauer character ψ. Then the Brauer character ψ^G of the induced K-representation U^G of G is given by

$$\psi^G(x) = [G:H]^{-1} \sum_{y \in G} \psi(y^{-1}xy), \qquad x \in G,$$

Received by the editors April 21, 1963.

[1] This research was supported by the Guggenheim Foundation and the Office of Naval Research.

where $\dot{\psi}$ coincides with ψ on H and vanishes outside of H. (See [**3**, §25].)

2. Our proof of Berman's theorem depends on several straightforward lemmas.

LEMMA 1. *Let $\phi_1, \cdots, \phi_s$ be the Brauer characters of a full set of irreducible K-representations $F_1, \cdots, F_s$ of G. Then $\phi_1, \cdots, \phi_s$ are linearly independent over $Q(\omega)$.*

PROOF. By [**4**, (70.24)], the field $K(\delta)$ is a splitting field for G, and so each ϕ_i is a sum (with non-negative integral coefficients) of the Brauer characters $\lambda_1, \cdots, \lambda_k$ introduced above. From [**4**, (29.6) and (69.4)] it follows that for $i \neq j$, ϕ_i and ϕ_j have no summands in common. Since the λ's are linearly independent over $Q(\omega)$, so are the ϕ's. This completes the proof.

LEMMA 2. *Let ϕ be the Brauer character of a K-representation F of G. If $a, b \in G$ are K-conjugate p-regular elements, then $\phi(a) = \phi(b)$.*

PROOF. Let $b^t = x^{-1}ax$, $x \in G$, $t \in T$. Then

$$F(x)^{-1}F(a)F(x) = (F(b))^t,$$

so that $F(a)$ and $(F(b))^t$ have the same characteristic roots. But the characteristic roots of $(F(b))^t$ are the tth powers of those of $F(b)$. Since each characteristic root of $F(b)$ is a power of δ, and since $\delta \rightarrow \delta^t$ is an automorphism of $K(\delta)$ which leaves $F(b)$ unchanged, we conclude that $F(b)$ and $(F(b))^t$ have the same characteristic roots. The lemma is thus established.

LEMMA 3. *Let $a \in G$ be a p-regular element of G of order m, and let $[a]$ denote the cyclic group it generates. Define θ by*

$$\theta(a^t) = \begin{cases} m, & t \in T, \\ 0, & t \notin T. \end{cases}$$

Then θ is a linear combination, with coefficients from $Q(\omega)$, of Brauer characters of K-representations of $[a]$.

PROOF. (Identical with the first paragraph of the proof of (42.5) in [**4**]. See also Berman [**1**], [**2**].)

3. We now prove Berman's theorem. Let $a_1, \cdots, a_r$ be representatives of the p-regular K-conjugacy classes in G, and let $\phi_1, \cdots, \phi_s$ be the Brauer characters of a full set of irreducible K-representations of G. By Lemma 2, each ϕ_i is completely determined by the r-tuple $(\phi_i(a_1), \cdots, \phi_i(a_r))$. By Lemma 1, these s r-tuples correspond-

ing to $\phi_1, \cdots, \phi_s$, are linearly independent over $Q(\omega)$. Therefore $s \leqq r$.

On the other hand, for each a_i we construct the function θ_i defined as in Lemma 3. Then $\theta_1^G, \cdots, \theta_r^G$ are linear combinations of $\phi_1, \cdots, \phi_s$ with coefficients from $Q(\omega)$. If we show that $\theta_1^G, \cdots, \theta_r^G$ are linearly independent over $Q(\omega)$, it will follow that $r \leqq s$, and we will be finished.

Let us compute $\theta_i^G(a_j)$. We have

$$\theta_i^G(a_j) = [G:H]^{-1} \sum_{y \in G} \dot{\theta}_i(y^{-1}a_j y).$$

When $j \neq i$, this gives $\theta_i^G(a_j) = 0$; for if $y^{-1}a_j y = a_i^t$ for some $y \in G$ and some $t \in T$, then a_i and a_j would be K-conjugate. On the other hand, in the expression for $\theta_i^G(a_i)$ the nonzero term $\dot{\theta}_i(a_i)$ always occurs, so that $\theta_i^G(a_i) \neq 0$. This completes the proof of the theorem.

We remark that the above proof is equally valid for $p = 0$.

References

1. S. D. Berman, *p-adic ring of characters*, Dokl. Akad. Nauk SSSR **106** (1956), 583–586.

2. ———, *The number of irreducible representations of a finite group over an arbitrary field*, Dokl. Akad. Nauk SSSR **106** (1956), 767–769.

3. R. Brauer and C. Nesbitt, *On the modular characters of groups*, Ann. of Math. **42** (1941), 556–590.

4. C. W. Curtis and I. Reiner, *Representation theory of finite groups and associative algebras*, Wiley, New York, 1962.

University of Illinois