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NOTE ON POINTWISE PERIODIC SEMIGROUPS

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An element x in a semigroup S is said to be periodic if there exists a positive integer n such that $x^{n+1}=x$, and the least such n, $p(x)$, is the period of x. S is pointwise periodic if each x in S is periodic. In [4], A. D. Wallace asks the following question concerning pointwise periodic topological semigroups.

Problem 3: If S is a pointwise periodic semigroup and is topologically an n-cell, is it possible that $S\backslash E$ is nonempty and $p(x)>2$ and constant on $S\backslash E$?

It will be shown that in a slightly more general situation than that of the above problem, it necessarily follows that $p(x)=2$ on $S\backslash E$.

The following notation will be used throughout this paper. For a semigroup S, $E=\{x: x\in S, x^2=x\}$ and for $e\in E$, $H(e)$ is the maximal subgroup of S containing the idempotent e. $H=\bigcup\{H(e): e\in E\}$ and functions y and θ are defined as in [5], that is, for $x\in H$, $y(x)$ is the idempotent of the unique maximal subgroup to which x belongs and $\theta(x)$ is the inverse of x in this group.

The following theorem will be proved:

Theorem. *Let S be a compact semigroup with the properties:*

(1) $S=H$,

(2) *for $e\in E$, $H(e)$ is totally disconnected,*

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(3) *there exists a closed subset B of S such that:*

(a) *$S\backslash B$ is a connected manifold dense in S,*

(b) $\dim B < \dim S$.

Then S is a pointwise periodic semigroup and $p(x) \leqq 2$ for each x in S.

Before proving this theorem the following lemma will be proved.

LEMMA. *Let S be a compact semigroup such that $S=H$ and for each e in E, $H(e)$ is totally disconnected. Then* $\dim S = \dim E$.

PROOF. Let the equivalence relation $\mathfrak{H}$ be defined as in [**5**]. Since $S=H$, each $\mathfrak{H}$-class is a group and $S/\mathfrak{H}$ is homeomorphic to E by the restriction to E of the canonical map of S onto $S/\mathfrak{H}$. By Anderson and Hunter [**1**, Lemma 5, p. 254], it follows that

$$\dim S \leqq \dim S/\mathfrak{H} + \max\{\dim H(e) : e \in E\}.$$

By assumption each $H(e)$ is totally disconnected, hence zero dimensional. Thus $\dim S \leqq \dim S/\mathfrak{H} = \dim E$ and since $E \subset S$, it follows that $\dim S = \dim E$.

PROOF OF THEOREM. If it can be shown that $\theta(x) = x$ for each x in $S\backslash B$, then it would follow that $x^3 = x(\theta(x)x) = xy(x) = x$ for $x \in S\backslash B$. But $T = \{x : x^3 = x\}$ is a closed subset of S and if $S\backslash B \subset T$, then $S = T$ since $S\backslash B$ is assumed to be dense in S. Thus it suffices to show that $\theta(x) = x$ for x in $S\backslash B$.

To prove $\theta | S\backslash B$ is the identity, let $A = \theta(S\backslash B) \cup S\backslash B$ and let $\theta_0 = \theta | A$. Since $\theta^2 = \theta$ and θ is a homeomorphism of S onto S [**5**], $\theta_0(A) = A$ is an open subset of S. Now let $E_0 = E \cap S\backslash B$. By the above lemma, $\dim S = \dim E$ which implies that $\dim E_0 = \dim S = \dim A$, since $\dim B < \dim S$. Hence E_0 is a nonempty subset of $S\backslash B$, $\theta_0(E_0) = E_0$ so that $E_0 \subset S\backslash B \cap \theta(S\backslash B)$ and A is connected. It will now be shown that F, the fixed point set of θ_0, has an interior point. From above, E_0 is a subset of the manifold A and $\dim E_0 = \dim A$ so that E_0 contains an interior point. Since $E_0 \subset F$, F also has an interior point. Because A is a connected manifold, θ_0 is a periodic homeomorphism and F has an interior point, it follows from a theorem of Montgomery and Zippin [**3**, Theorem 1, p. 223] that θ_0 is the identity map. This completes the proof of the theorem.

COROLLARY. *Let S be a pointwise periodic semigroup on an n-cell. Then $p(x) \leqq 2$ for all x in S.*

PROOF. A pointwise periodic semigroup is the union of groups [**2**, Theorem 1.9, p. 20], hence $S=H$. Also $H(e)$ is totally disconnected for each e in E since $H(e)$ is a compact periodic group. Letting B, in the theorem, be the bounding $(n-1)$-sphere of S, the corollary follows immediately.

EXAMPLE (WALLACE [6]). Let $I=[0, 1]$ be the unit interval with multiplication $xy=\min\{x, y\}$. Let $G=\{e, a\}$ be the two element group with identity e and let $T_0=G\times I$ with coordinatewise multiplication. Then $K_0=G\times\{0\}$ is an ideal of T_0 and $T=T_0/K_0$, the Rees quotient of T_0 by the ideal K_0 is a pointwise periodic semigroup whose topological space is a 1-cell.

T^n, the n-fold cartesian product of T, with coordinatewise multiplication is a pointwise periodic semigroup on an n-cell.

The above example may be varied by letting one of the intervals in the cartesian product be a semigroup with multiplication $xy=x$. In this manner one obtains a pointwise periodic semigroup without a two-sided identity.

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