COMMUTING BOOLEAN ALGEBRAS OF PROJECTIONS. II. BOUNDEDNESS IN L_p

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This paper complements our previous paper of the same title [4]. The conclusions, results applicable to the theory of spectral operators, are those of the previous paper, but the hypotheses are disjoint and the methods are somewhat different; we will make only historical reference to this work.

One of the basic problems of the theory of spectral operators is whether the sum and product of two commuting spectral operators on a Banach space is again spectral (for background material on spectral operators see especially [1] or [2]). Wermer [5] showed that this is in fact always the case if the operators act on a Hilbert space. Dunford [1] and Foguel [3] proved that if the underlying space is weakly complete, the boundedness of the Boolean algebra of projections generated by the resolutions of the identity of the operators implies that the sum and product is spectral. In practice, however, this may be difficult to determine, and our work has been to find easily applicable criteria for this boundedness. Our previous paper [4] gave a criterion in terms of multiplicity: It sufficed that one of the algebras of projections was of finite multiplicity, and that even for some separable reflexive Banach spaces this condition was necessary. Our present criterion is in terms of the underlying space and independent of various properties the generating Boolean algebras might enjoy other than their boundedness. Our result holds for L_p spaces, $1<p<\infty$, direct sums of L_p spaces for which the p's are bounded away from 1 and ∞, and for subspaces thereof; especially of interest for partial differential equations is that our theorem holds for the Sobolev spaces where the norm of a function x is given by the sum of the L_p norm of x and of some perhaps different L_p norms of its derivatives. Since our estimates are all finite-combinatoric, our results hold for inseparable L_p spaces, L_p spaces with respect to only finitely additive measures, and their elaborations.

Our first section will establish some combinatorial propositions, which when used in the second section will give our theorem for $2\leqq p<\infty$; consideration of adjoints gives the theorem for $1<p\leqq 2$.

Received by the editors December 29, 1962, and, in revised form, June 13, 1963.

[1] Supported by NSF G-14137.

1. Averaging propositions. We shall wish to consider the average value of such expressions as $\left|\sum_j c_j\mu_j\right|^p$ as the c_j vary independently over complex numbers of absolute value one; to be explicit, we define such things as $\text{Average}_{|c_j|=1}\left|\sum_{j=1}^n c_j\mu_j\right|^p$ to be

$$(2\pi)^{-n}\int_0^{2\pi} d\theta_1, \cdots, \int_0^{2\pi} d\theta_n\left|\sum_{j=1}^n e^{i\theta_j}\mu_j\right|^p.$$

PROPOSITION 1. *Let $\mu_1 \cdots \mu_n$ be any complex numbers, and $p \geqq 2$. Then*

$$\left(\sum_j |\mu_j|^2\right)^{p/2} \leqq \underset{|c_j|=1}{\text{Average}}\left|\sum_j c_j\mu_j\right|^p \leqq \Gamma\left(\frac{p}{2}+2\right)\left(\sum_j |\mu_j|^2\right)^{p/2}.$$

PROOF. First assume p is an even integer, $p=2N$. Then

$$\begin{aligned}\left|\sum_j c_j\mu_j\right|^{2N} &= \left(\sum_{j,k} c_j\bar{c}_k\mu_j\bar{\mu}_k\right)^N \\ &= \sum_{\substack{j_1,\cdots,j_N \\ k_1,\cdots,k_N}} c_{j_1}\cdots c_{j_N}\bar{c}_{k_1}\cdots\bar{c}_{k_N}\mu_{j_1}\cdots\mu_{j_N}\bar{\mu}_{k_1}\cdots\bar{\mu}_{k_N}.\end{aligned}$$

Now when we perform the integrations to compute the average, all terms vanish except those for which $k_1, \cdots, k_N$ is a permutation of $j_1, \cdots, j_N$; and such a term contributes $|\mu_{j_1}|^2 \cdots |\mu_{j_N}|^2$ to the average for each distinct permutation of the j's. If a given set $j_1, \cdots, j_N$ consists of r integers with multiplicities $m_1, \cdots, m_r$, then the number of distinct permutations is the N-nomial coefficient $N!/m_1!m_2! \cdots m_r!$ which in any case lies between 1 and $N!$.

Thus the average lies between

$$\sum_{j_1,\cdots,j_N} |\mu_{j_1}|^2 \cdots |\mu_{j_N}|^2 = (\sum |\mu_j|^2)^N \quad \text{and} \quad N!\left(\sum_j |\mu_j|^2\right)^N,$$

proving the proposition in this case.

For other p, set $p=2(N+\epsilon)$, where N is an integer and $0<\epsilon<1$. Hölder's inequality, applied to the integrals defining the average, yields an upper bound:

$$\begin{aligned}\underset{|c_j|=1}{\text{Average}}\left|\sum_j c_j\mu_j\right|^{2(N+\epsilon)} &\leqq \left(\underset{|c_j|=1}{\text{Average}}\left|\sum_j c_j\mu_j\right|^{2(N+1)}\right)^{(N+\epsilon)/(N+1)} \\ &\leqq \Gamma(N+2)\left(\sum_j |\mu_j|^2\right)^{N+\epsilon};\end{aligned}$$

and the lower bound:

$$\left(\sum_j |\mu_j|^2\right)^{(N+\epsilon)} \leqq \left(\operatorname*{Average}_{|c_j|=1}\left|\sum_j c_j\mu_j\right|^{2N}\right)^{(N+\epsilon)/N}$$
$$\leqq \operatorname*{Average}_{|c_j|=1}\left|\sum_j c_j\mu_j\right|^{2(N+\epsilon)}.$$

PROPOSITION 2. *Let* λ_{jk}, $j=1, \cdots, n$, $k=1, \cdots, m$ *be any complex numbers and let* $p \geqq 2$. *Then*

$$\left(\sum_{jk} |\lambda_{jk}|^2\right)^{p/2} \leqq \operatorname*{Average}_{|d_k|=1}\left(\sum_j\left|\sum_k d_k\lambda_{jk}\right|^2\right)^{p/2}$$
$$\leqq \Gamma\left(\frac{p}{2}+2\right)\left(\sum_{jk}|\lambda_{jk}|^2\right)^{p/2}.$$

PROOF. We first obtain the upper bound. Let p be an even integer, $p=2N$. We have

$$\left(\sum_j\left|\sum_k d_k\lambda_{jk}\right|^2\right)^N$$
$$= \sum_{\substack{k_1,\cdots,k_N\\ l_1,\cdots,l_N}} d_{k_1}\cdots d_{k_N}\bar d_{l_1}\cdots \bar d_{l_N}\left(\sum_j \lambda_{jk_1}\bar\lambda_{jl_1}\right)\cdots\left(\sum_j \lambda_{jk_N}\bar\lambda_{jl_N}\right).$$

As before, the only nonzero contributions after performing the integrations to compute the average are those for which $l_1, \cdots, l_N$ is a permutation of $k_1, \cdots, k_N$. For any such permutation, the contribution is bounded in absolute value by

$$\left|\sum_j \lambda_{jk_1}\bar\lambda_{jl_1}\right|\cdots\left|\sum_j \lambda_{jk_N}\bar\lambda_{jl_N}\right|$$
$$\leqq \left(\sum_j|\lambda_{jk_1}|^2\right)^{1/2}\left(\sum_j|\lambda_{jl_1}|^2\right)^{1/2}\cdots\left(\sum_j|\lambda_{jk_N}|^2\right)^{1/2}\left(\sum_j|\lambda_{jl_N}|^2\right)^{1/2}$$
$$= \left(\sum_j|\lambda_{jk_1}|^2\right)\cdots\left(\sum_j|\lambda_{jk_N}|^2\right).$$

Since there are at most $N!$ such permutations, the average is bounded above by

$$N! \sum_{k_1,\cdots,k_N}\left(\sum_j|\lambda_{jk_1}|^2\right)\cdots\left(\sum_j|\lambda_{jk_N}|^2\right) = N!\left(\sum_{j,k}|\lambda_{jk}|^2\right)^N.$$

The upper bound for general p is obtained from the Hölder inequality exactly as in Proposition 1.

For the lower bound, we consider first $p=2$:

$$\operatorname*{Average}_{|d_k|=1} \sum_j \left| \sum_k d_k \lambda_{jk} \right|^2 = \operatorname*{Average} \sum_{k,l} d_k \bar{d}_l \sum_j \lambda_{jk} \bar{\lambda}_{jl}$$
$$= \sum_k \sum_j \lambda_{jk} \bar{\lambda}_{jk} = \sum_{jk} |\lambda_{jk}|^2.$$

For any other p, we Hölder:

$$\left(\sum_{j,k} |\lambda_{jk}|^2 \right)^{p/2} = \left(\operatorname*{Average}_{|d_k|=1} \sum_j \left| \sum_k d_k \lambda_{jk} \right|^2 \right)^{p/2}$$
$$\leqq \operatorname*{Average}_{|d_k|=1} \left(\sum_j \left| \sum_k d_k \lambda_{jk} \right|^2 \right)^{p/2}.$$

Propositions 1 and 2 together give the inequality we wish, namely,

$$\left(\sum_{j,k} |\lambda_{jk}|^2 \right)^{p/2} \leqq \operatorname*{Average}_{|c_j|=1, |d_k|=1} \left| \sum_{j,k} c_j d_k \lambda_{jk} \right|^p$$
$$\leqq \Gamma\left(\frac{p}{2}+2\right)^2 \left(\sum_{j,k} |\lambda_{jk}|^2 \right)^{p/2}$$

for any complex numbers λ_{jk}, and $p \geqq 2$.

2. **The boundedness theorem.** Let $\mathcal{E}$ be a Boolean algebra of projections on a Banach space. A theorem of Dunford [1, Theorem 7] states that if $E_1, \cdots, E_n$ are mutually disjoint members of $\mathcal{E}$, and if $a_1, \cdots, a_n$ are complex numbers, then the norm of the operator $\sum_{j=1}^n a_j E_j$ is at most $4 \cdot \max_j |a_j| \sup_{E \in \mathcal{E}} \|E\|$. We define $\|\mathcal{E}\|$ to be $\sup \{ \| \sum_j a_j E_j \| : |a_j| \leqq 1, E_j$ disjoint projections in $\mathcal{E} \}$ and note that $\|\mathcal{E}\|$ is finite, if and only if there is a finite upper bound to $\|E\|$, $E \in \mathcal{E}$.

THE BOUNDEDNESS THEOREM. *Let $\mathcal{E}$ and $\mathcal{F}$ be two bounded commuting Boolean algebras of projections on L_p, and let $\mathcal{G}$ be the Boolean algebra of projections consisting of finite sums of disjoint projections EF, $E \in \mathcal{E}$, $F \in \mathcal{F}$. Then*

$$\|\mathcal{G}\| \leqq \left(\Gamma\left(\frac{p}{2} + 2 \right) \right)^{2/p} \|\mathcal{E}\|^2 \|\mathcal{F}\|^2$$

if $p \geqq 2$, or the same quantity with p replaced by $p/(p-1)$ if $p<2$.

PROOF. If $\mathcal{E}^* = \{E^* : E \in \mathcal{E}\}$, then the Boolean algebra $\mathcal{E}^*$ satisfies $\|\mathcal{E}\| = \|\mathcal{E}^*\|$, so that it suffices to consider, of L_p and $L_{p/(p-1)}$, only that one of larger exponent.

When we compute $\|\mathcal{G}\|$ it suffices to consider operators of the form

$\sum_{j=1}^n \sum_{k=1}^m a_{jk}E_jF_k$ with $|a_{jk}|=1$ and $\sum_j E_j=\sum_k F_k=I$, the identity on L_p; for any operator which we use to compute $\|\mathcal{G}\|$ is in the convex hull of such operators. Now let x be a function in L_p. We wish to compute $\|\sum_{jk} a_{jk}E_jF_kx\|^p$. For any choice of $c_1, \cdots, c_n$ each of absolute value 1, the operator $\sum_j c_jE_j$, as well as its inverse $\sum_j \bar{c}_jE_j$, has bound $\|\mathcal{E}\|$. The analogous holds true for an operator $\sum_k d_kF_k$ where $d_1, \cdots, d_m$ are each of absolute value 1. Thus

$$\|\mathcal{E}\|^{-p}\|\mathcal{F}\|^{-p}\left\|\sum_{j,k} c_jd_ka_{jk}E_jF_kx\right\|^p \leqq \left\|\sum_{j,k} a_{jk}E_jF_kx\right\|^p \leqq \|\mathcal{E}\|^p\|\mathcal{F}\|^p\left\|\sum_{j,k} c_jd_ka_{jk}E_jF_kx\right\|^p.$$

If we average over all $|c_j|=1$, $|d_k|=1$, the central term of this inequality remains unchanged. However,

$$\operatorname*{Average}_{|c_j|=1,|d_k|=1}\left\|\sum_{j,k} c_jd_ka_{jk}E_jF_kx\right\|^p = \operatorname*{Average}_{|c_j|=1,|d_k|=1}\int\left|\sum_{j,k} c_jd_ka_{jk}E_jF_kx\right|^p = \int \operatorname*{Average}_{|c_j|=1,|d_k|=1}\left|\sum_{j,k} c_jd_ka_{jk}E_jF_kx\right|^p.$$

By Propositions 1 and 2, this is bounded below by

$$\int\left(\sum_{j,k} |a_{jk}E_jF_kx|^2\right)^{p/2} = \int\left(\sum_{j,k} |E_jF_kx|^2\right)^{p/2}$$

(remember that $|a_{jk}|=1$), and above by

$$\Gamma\left(\frac{p}{2}+2\right)^2\cdot\int\left(\sum_{j,k} |E_jF_kx|^2\right)^{p/2}.$$

We thus have

$$\begin{aligned}\left\|\sum_{j,k} a_{jk}E_jF_kx\right\|^p &\leqq \|\mathcal{E}\|^p\|\mathcal{F}\|^p\cdot\Gamma\left(\frac{p}{2}+2\right)^2\int\left(\sum_{j,k} |E_jF_kx|^2\right)^{p/2} \\ &\leqq \|\mathcal{E}\|^p\|\mathcal{F}\|^p\Gamma\left(\frac{p}{2}+2\right)^2 \operatorname*{Average}_{|c_j|=1,|d_k|=1}\left\|\sum_{j,k} c_jd_kE_jF_kx\right\|^p \\ &\leqq \|\mathcal{E}\|^{2p}\|\mathcal{F}\|^{2p}\Gamma\left(\frac{p}{2}+2\right)^2\|x\|^p.\end{aligned}$$

Therefore $\|\mathcal{G}\| \leqq \Gamma(p/2+2)^{2/p}\|\mathcal{E}\|^2\|\mathcal{F}\|^2$.

We wish to make the remark that the only assumptions about the underlying Banach space was that it was a space of functions x in which the norm was defined as $(\int|x|^p)^{1/p}$. The E's and F's may be such that they only operate on a certain subspace, as is the case with the Sobolev spaces which can be regarded as a subspace of a sum of L_p spaces; thus, our theorem holds in such spaces, and of course for others as mentioned in the introduction.

3. **Further remarks.** If $E_1, \cdots, E_n$ are the atoms of a finite Boolean algebra $\mathcal{E}$ of projections on L_2, then an equivalent norm $|||\cdot|||$ on L_2 is given by

$$\|\mathcal{E}\|^{-2}\|x\|^2 \leqq |||x|||^2 = \sum_j \|E_jx\|^2 \leqq \|\mathcal{E}\|^2\|x\|^2.$$

Now if $\mathfrak{F}$ is a bounded commuting algebra of projections, we have

$$\left|\left|\left|\sum_j E_jF_jx\right|\right|\right|^2 = \sum_j \|E_jF_jx\|^2 \leqq \|\mathfrak{F}\|^2|||x|||^2$$

so that if $\mathcal{G} = \{\sum_j E_jF_j;\ E_j \text{ disjoint in } \mathcal{E},\ F_j \in \mathfrak{F}\}$, then $\|\mathcal{G}\| \leqq 4\|\mathcal{E}\|^2\|\mathfrak{F}\|$. This is essentially Wermer's method. It is natural to ask whether in general $(\sum_j \|E_jx\|^p)^{1/p}$ is an equivalent norm in L_p, the equivalence depending only upon $\|\mathcal{E}\|$; for then the boundedness theorem would be very simple. We are indebted to E. R. Rodemich for the following counterexample in L_4:

In $L_4(0, 1)$ consider the functions $\phi_j(t) = (-1)^{a_j(t)}$, $0<t<1$, $1 \leqq j < \infty$, where t has the dyadic expansion $t = \sum_{j=1}^\infty a_j(t)2^{-j}$ and each $a_j(t)$ is either 0 or 1 (a_j is well defined except on a set of measure 0; the ϕ_j are Rademacher functions). Let the operator E_j be defined as carrying the function x into $E_jx = (\int x\phi_j)\phi_j$. $\int \phi_j\phi_k = \delta_{jk}$ shows that the operators E_j are a disjoint set of projections. Further if we compute the norm of the operator $\sum_{j=1}^n a_jE_j$ ($|a_j| \leqq 1$), we have, setting $b_j = a_j \int x\phi_j$,

$$\begin{aligned}\left\|\sum_j a_jE_jx\right\|^4 &= \int\left|\sum_j b_j\phi_j\right|^4 \\ &= \int \sum_{j,k,l,m} b_jb_k\bar{b}_l\bar{b}_m\phi_j\phi_k\phi_l\phi_m \\ &= \sum_j |b_j|^4 + \sum_{j,k}(b_j\bar{b}_k)^2 + 2\sum_{j,k}|b_jb_k|^2 \\ &\leqq 4\left(\sum_j |b_j|^2\right)^2.\end{aligned}$$

(We have used $\int\phi_j\phi_k\phi_l\phi_m = 0$ or 1, and 1 only in the cases $j=k=l=m$, $j=k\neq l=m$, $j=l\neq k=m$, and $j=m\neq k=l$.)

Since the functions ϕ_j are orthonormal in $L_2(0, 1)$, $\sum_j |b_j|^2 = \sum_j |\int x\phi_j|^2 \leqq \int |x|^2 \leqq \|x\|_4^2$; thus the norm of the operator $\sum_j a_jE_j$ ($|a_j| \leqq 1$) is at most $4^{1/4}$. Therefore the Boolean algebras $\mathcal{E}_n$ whose atoms are $E_1, \cdots, E_n, E_0^{(n)} = I - \sum_1^n E_j$, satisfy $\|\mathcal{E}_n\| < 3$, since every member of $\mathcal{E}_n$ is a finite sum of E_j's or I minus such a finite sum.

Now let $x_n = \phi_1 + \cdots + \phi_n$. $E_jx_n = \phi_j$ $(0 < j \leqq n)$, and $E_jx_n = 0$ $(j > n)$. The L_4 norm of x_n has its fourth power equal to $n + 3n^2$. However, if E_j runs over all atoms of $\mathcal{E}_n$, $\sum_j \|E_jx_n\|^4 = \sum_{j=1}^n \|E_jx_n\|^4 = \sum_{j=1}^n \|\phi_j\|^4 = n$ which is not comparable with $\|x_n\|^4$ uniformly in n.

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