ON THE NEWTON POLYTOPE

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1. **Introduction.** The theory of the Newton polygon of a polynomial in one variable with coefficients in a complete non-Archimedean valued field is well known (see, for example, [1], [2], [3], [6]). In [4], Krasner states that one may construct an analogous Newton polytope for a polynomial in several variables. In this paper we explore the properties of the Newton polytope.

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2. **Preliminaries.** Let K be a complete field with respect to a non-Archimedean rank one valuation $x \to \text{ord}\, x$ with value group $\mathfrak{G} \subset R$, where R denotes the additive group of real numbers. We shall assume that $\mathfrak{G}$ is dense in R. Let $\mathfrak{K}$ be the algebraic closure of K, and extend the valuation to $\mathfrak{K}$ in the natural manner. As in [2], for each real number b we define $\Gamma_b = \{\xi \in \mathfrak{K} : \text{ord}\, \xi = b\}$.

DEFINITION 1. Let $f(x) = \sum_{i=0}^{n} a_i x^i \in K[x]$. For any $\mu \in R$, $v(f;\, \mu) = \text{Min}_{0 \leq i \leq n} (\text{ord}\, a_i + i\mu)$.

REMARK. $v(f;\, \mu)$ is the Y-intercept of the lower line of support of the Newton polygon of f with slope $-\mu$.

We need the following results from the one-variable theory.

PROPOSITION 1. *Let $f(x) \in K[x]$ have a zero on Γ_r. Then for any $\lambda \in \mathfrak{G}$ satisfying the inequality $\lambda \geq v(f;\, r)$, there exists $\xi \in \Gamma_r$ such that* ord $f(\xi) = \lambda$.

PROOF. (a) If $a_0 \neq 0$ and $-r$ is the slope of the first side of the Newton polygon of f (i.e., if, for all $r' > r$, f has no zero on $\Gamma_{r'}$) then clearly $v(f;\, r) = \text{ord}\, a_0$. Therefore, we need only choose $\gamma \in \Gamma_\lambda$ such that $\text{ord}\,(a_0 - \gamma) = \text{ord}\, a_0$, for then the polynomials $f(x)$ and $f(x) - \gamma$ will have identical Newton polygons. If $\lambda > v(f;\, r)$, then for any $\gamma \in \Gamma_\lambda$, $\text{ord}\,(a_0 - \gamma) = \text{ord}\, a_0$; if $\lambda = v(f;\, r)$, we choose $\alpha,\ \beta \in \Gamma_0$ such that $\alpha + \beta \in \Gamma_0$ (this can be done since the residue class field of $\mathfrak{K}$ contains more than two elements), and put $\gamma = a_0(1 + \alpha\beta^{-1})$.

(b) If either $a_0 = 0$ or $-r$ is not the slope of the first side of the Newton polygon of f, let γ be any element of Γ_λ, and consider the Newton diagram of $f(x) - \gamma$: clearly the Newton diagram of $f(x) - \gamma$ coincides with the Newton diagram of $f(x)$, with the possible excep-

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tion of the points with zero abscissa. Since $\operatorname{ord}(a_0-\gamma)\geqq v(f; r)$, $-r$ is the slope of a side of the Newton polygon of $f(x)-\gamma$.

The following result is essentially identical to Lemma 1.2 of [2].

LEMMA 1. *Let* $f_1(x), f_2(x), \cdots, f_n(x)$ *be a finite set of polynomials with coefficients in* K, *let* $\rho\in\mathfrak{G}$. *Then there exists* $\xi\in\Gamma_\rho$ *such that* $\operatorname{ord} f_i(\xi)=v(f_i; \rho)$, $i=1, 2, \cdots, n$.

PROOF. Let $v(f_i; \rho)=M_i$, $i=1, 2, \cdots, n$; then $M_i\in\mathfrak{G}$. Therefore, we may choose $\pi_i\in\Gamma_{M_i}$, $\pi\in\Gamma_\rho$. For each i, we put $g_i(x)=f_i(\pi x)/\pi_i$. Then the coefficients of $g_i(x)$ are integral and the image of $g_i(x)$ in the residue class field of $\mathfrak{K}$ is nontrivial. Since the residue class field is infinite there is a unit ξ' in $\mathfrak{K}$ such that $\operatorname{ord} g_i(\xi')=0$, $i=1, 2, \cdots, n$. If we put $\xi=\pi\xi'$, we have the desired result.

3. **The Newton polytope.** Let $f(x, y)=\sum a_{ij}x^iy^j\in K[x, y]$. The point set $\{(i, j, \operatorname{ord} a_{ij})\}$ is called the *Newton diagram* of $f(x, y)$. We define the convex closure of the Newton diagram of $f(x, y)$ with the point $(0, 0, +\infty)$ to be the *Newton polytope* of $f(x, y)$.

REMARK. The Newton polytope of $f(x, y)$ is the graph of the function

$$\Pi_f(X, Y) = \sup_{\mu,\nu\in R} [v(f; \mu, \nu) - \mu X - \nu Y],$$

where $v(f; \mu, \nu)$ is defined in the obvious manner generalizing Definition 1: $v(f; \mu, \nu)=\operatorname{Min}_{i,j}(\operatorname{ord} a_{ij}+i\mu+j\nu)$ (see [5, p. 49]).

Let $(\xi, \eta)\in\mathfrak{K}\times\mathfrak{K}$, suppose $(\xi, \eta)\in\Gamma_\rho\times\Gamma_\sigma$. The following result gives an estimate for $\operatorname{ord} f(\xi, \eta)$ in terms of ρ, σ.

PROPOSITION 2. *Let* P *be the lower plane of support of the Newton polytope of* $f(x, y)$, *with* $\partial Z/\partial X=-\rho$, $\partial Z/\partial Y=-\sigma$. *Suppose* $(\xi, \eta)\in\Gamma_\rho\times\Gamma_\sigma$. *If only one vertex of the polytope lies on* P, *then only one term of* $f(\xi, \eta)$ *attains minimal* ord, *and then* $\operatorname{ord} f(\xi, \eta)=v(f; \rho, \sigma)$, *the* Z*-axis intercept of* P. *Otherwise*, $\operatorname{ord} f(\xi, \eta)\geqq v(f; \rho, \sigma)$.

PROOF. Let the plane P_{ij} be defined by the equation $Z+\rho X+\sigma Y=\operatorname{ord}(a_{ij}\xi^i\eta^j)$. Then the point $(i, j, \operatorname{ord} a_{ij})$ in the Newton diagram of $f(x, y)$ lies in P_{ij}; but $\operatorname{ord}(a_{ij}\xi^i\eta^j)<\operatorname{ord}(a_{i'j'}\xi^{i'}\eta^{j'})$ (respectively $\operatorname{ord}(a_{ij}\xi^i\eta^j)\leqq\operatorname{ord}(a_{i'j'}\xi^{i'}\eta^{j'})$) if and only if the intercept cut off on the Z-axis by the plane P_{ij} is less than (respectively less than or equal to) that cut off by $P_{i'j'}$. Thus, $\operatorname{ord}(a_{i_0j_0}\xi^{i_0}\eta^{j_0})=\operatorname{Min}_{i,j}\operatorname{ord}(a_{ij}\xi^i\eta^j)$ if and only if $P_{i_0j_0}$ is the lower plane of support of the Newton polytope with $\partial Z/\partial X=-\operatorname{ord}\xi$, $\partial Z/\partial Y=-\operatorname{ord}\eta$.

COROLLARY. *If* (ξ, η) *is a zero of* $f(x, y)$, *then the lower plane of sup-*

port P of the Newton polytope of $f(x, y)$ with $\partial Z/\partial X = -\text{ord}\,\xi$, $\partial Z/\partial Y = -\text{ord}\,\eta$ contains an edge of the polytope.

REMARK. No distinction is made here between the plane P containing an edge or a face of the polytope.

The converse to the corollary of Proposition 2 is also valid. Thus, the Newton polytope of $f(x, y)$ gives an explicit criterion for determining the existence of a zero of $f(x, y)$ on $\Gamma_r \times \Gamma_s$. Before proceeding to the proof of the converse, we introduce the following notation.

Let $f(x, y) = f_0(x) + f_1(x)y + f_2(x)y^2 + \cdots + f_n(x)y^n$, $f_i(x) \in K[x]$, $i = 0, 1, 2, \cdots, n$. We shall assume that $f(x, y) \notin K[x]$, $f_n(x) \neq 0$. Let Π denote the Newton polytope of $f(x, y)$. For $\rho \in \mathfrak{G}$, let Λ_ρ be the convex closure in the YZ-plane of the point set $\{(0, j, v(f_j; \rho)) : j = 0, 1, 2, \cdots, n\}$ with the point $(0, 0, +\infty)$. For $\xi \in K$, let Λ_ξ be the convex closure in the YZ-plane of the point set $\{(0, j, \text{ord}\, f_j(\xi)) : j = 0, 1, 2, \cdots, n\}$ with the point $(0, 0, +\infty)$. We observe that Λ_ξ is the Newton polygon of the polynomial $g_\xi(y) = \sum f_j(\xi) y^j$, and that if $\text{ord}\,\xi = \rho$, then no point of Λ_ρ lies below Λ_ξ. Let Π_j denote the Newton polygon of the polynomial $f_j(x)$ in the plane $Y = j$, and finally let $l_j(\rho)$ be the lower line of support of Π_j with slope $-\rho$ in the plane $Y = j$.

PROPOSITION 3. *Let $f(x, y) \in K[x, y]$, let $r, s \in \mathfrak{G}$. Suppose P_{rs} is the lower plane of support of Π, the Newton polytope of $f(x, y)$, with equation $Z + rX + sY + d = 0$. If P_{rs} contains an edge of Π, then there is a point $(\xi, \eta) \in \Gamma_r \times \Gamma_s$ such that $f(\xi, \eta) = 0$.*

PROOF. Suppose P_{rs} contains an edge of Π with direction numbers (α, β, γ). Since P_{rs} cannot contain a vertical line, either α or β is different from zero. We may assume, with no loss of generality, that $\beta \neq 0$. Then a pair of points $p_1 = (i_1, j_1, \text{ord}\, a_{i_1 j_1})$, $p_2 = (i_2, j_2, \text{ord}\, a_{i_2 j_2})$ of the Newton diagram of $f(x, y)$ is on P_{rs}, with $j_1 \neq j_2$. Since P_{rs} is a lower plane of support of Π containing p_1 and p_2, with $\partial Z/\partial X = -r$, it follows that $l_{j_1}(r)$ and $l_{j_2}(r)$ are in P_{rs}. By Lemma 1, we may choose $\xi \in \Gamma_r$ such that $\text{ord}\, f_{j_1}(\xi) = v(f_{j_1}; r)$, $\text{ord}\, f_{j_2}(\xi) = v(f_{j_2}; r)$. Thus, the points $q_1 = (0, j_1, \text{ord}\, f_{j_1}(\xi))$ and $q_2 = (0, j_2, \text{ord}\, f_{j_2}(\xi))$ of the Newton diagram of $g_\xi(y)$ are in P_{rs}, and are therefore on a side of Λ_ξ which lies in P_{rs} (since no point of Λ_ξ can lie below the intersection of P_{rs} with the YZ-plane). But since Λ_ξ lies in the $(X = 0)$-plane, we see that the side of Λ_ξ determined by q_1, q_2 has slope $\partial Z/\partial Y = -s$; therefore, the polynomial $g_\xi(y)$ has a root $\eta \in \Gamma_s$. Hence, $(\xi, \eta) \in \Gamma_r \times \Gamma_s$ and $f(\xi, \eta) = 0$.

We summarize these results in

THEOREM 1. *Let $f(x, y) \in K[x, y]$, let $r, s \in \mathfrak{G}$, and let P_{rs} be the lower plane of support of the Newton polytope of $f(x, y)$ with $\partial Z/\partial X = -r$, $\partial Z/\partial Y = -s$. There is a zero (ξ, η) of $f(x, y)$ such that* ord $\xi = -r$, ord $\eta = -s$ *if, and only if, the plane P_{rs} contains an edge of the polytope.*

4. Distinguished values.

DEFINITION 2. Let D be a subset of $\mathfrak{K} \times \mathfrak{K}$, let r (respectively s) be a real number. We say that r is *x-distinguished on D* (respectively, s is *y-distinguished on D*) if there are infinitely many $s \in \mathfrak{G}$ (respectively, infinitely many $r \in \mathfrak{G}$) such that $D \cap (\Gamma_r \times \Gamma_s) \neq \emptyset$.

PROPOSITION 4. *Let $f(x, y) \in K[x, y]$, suppose $f(x, y) \neq 0$; let $D = V(f) = \{(\xi, \eta) \in \mathfrak{K} \times \mathfrak{K} : f(\xi, \eta) = 0\}$. The set of real numbers which are x-distinguished on D (respectively, y-distinguished on D) is finite.*

PROOF. Let $f(x, y) = f_0(x) + f_1(x)y + f_2(x)y^2 + \cdots + f_n(x)y^n$, $f_i(x) \in K[x]$, $0 \leqq i \leqq n$. Since $f(x, y) \neq 0$, not all the polynomials $\{f_i(x)\}$ are zero. Let $\mathfrak{F}$ be the subset of $\{f_i(x) : 0 \leqq i \leqq n\}$ consisting of those polynomials which are nonzero, and let $\mathfrak{R}$ be the set of values of zeros of polynomials in $\mathfrak{F}$, i.e., $r \in \mathfrak{R}$ if there is a pair $(f, \xi) \in \mathfrak{F} \times \Gamma_r$ such that $f(\xi) = 0$. Clearly $\mathfrak{R}$ is a finite set. Suppose $r' \notin \mathfrak{R}$. Then for $\xi \in \Gamma_{r'}$, the Newton diagram of $g_\xi(y) = f(\xi, y)$ depends only on ord ξ. Therefore, as ξ runs through $\Gamma_{r'}$, there is only a finite number of $s \in \mathfrak{G}$ such that g_ξ has a zero on Γ_s. Therefore if $r' \notin \mathfrak{R}$, r' is not x-distinguished on D.

The set of real numbers which are distinguished for a given polynomial is determined by the Newton polytope of that polynomial. In fact, we shall prove

THEOREM 2. *Let $f(x, y) \in K[x, y]$, suppose $f(0, 0) \neq 0$. Then ρ is x-distinguished on $V(f)$ if, and only if, there is an edge of the Newton polytope of $f(x, y)$ with direction numbers* $(1, 0, -\rho)$.

The proof of Theorem 2 will be a trivial consequence of Propositions 5 and 6.

LEMMA 2. *Let $f(x, y) = f_0(x) + f_1(x)y + f_2(x)y^2 + \cdots + f_n(x)y^n \in K[x, y]$, suppose $f(x, y) \notin K[x]$, $f(0, 0) \neq 0$. Let $\rho \in \mathfrak{G}$, and let Λ_ρ, Π_j, $0 \leqq j \leqq n$, be as previously defined. If the point* $(0, j_0, v(f_{j_0}; \rho))$ *is on Λ_ρ, then there is a vertex* $(i_0, j_0, \text{ord } a_{i_0 j_0})$ *of Π_{j_0} which is on the Newton polytope of $f(x, y)$.*

PROOF. Suppose the point $(0, j_0, v(f_{j_0}; \rho))$ is on the side of Λ_ρ with vertices $(0, j_1, v(f_{j_1}; \rho))$, $(0, j_2, v(f_{j_2}; \rho))$, and suppose $j_1 < j_2$. Let P be the plane determined by the (parallel) lines $l_{j_1}(\rho)$ and $l_{j_2}(\rho)$. Then certainly $l_{j_0}(\rho)$ lies in P. It remains only to be shown that P is a lower plane of support of the polytope. Suppose not; then there is a point $(i', j', \text{ord } a_{i'j'})$ below P. Hence $(0, j', v(f_{j'}; \rho))$ lies below the line $P\cap(X=0)$. This contradicts convexity of Λ_ρ in the YZ-plane.

COROLLARY. *Using the above notation, if* $(0, j_0, v(f_{j_0}; \rho))$ *is on* Λ_ρ, *and if* ord $f_{j_0}(\xi)$ *has more than one value for* $\xi\in\Gamma_\rho$, *then a side of* Π_{j_0} *is on the polytope of* $f(x, y)$.

PROPOSITION 5. *If* $\rho\in\mathfrak{G}$ *is x-distinguished on* $V(f)$, *then there is an edge of the Newton polytope of f with direction numbers* $(1, 0, -\rho)$.

PROOF. For $\xi\in\Gamma_\rho$, we let $g_\xi(y)$, Λ_ξ, Λ_ρ be defined as before. Since ρ is x-distinguished on $V(f)$, the set of slopes of sides of the polygons $\{\Lambda_\xi : \xi\in\Gamma_\rho\}$ is infinite. Consider the set of non-negative integers j with the property that $(0, j, v(f_j; \rho))$ is a vertex of Λ_ρ and $\{\text{ord } f_j(\xi) : \xi\in\Gamma_\rho\}$ has more than one element. If this set were empty, it would follow that $\Lambda_\rho=\Lambda_\xi$ for each $\xi\in\Gamma_\rho$, contradicting the hypothesis. Let j_0 denote the smallest integer of this set.

By the previous corollary, $l_{j_0}(\rho)$ contains a side of Π_{j_0}, and this side is on $\mathbf{\Pi}$. To complete the proof of Proposition 5, we need only show that this side of Π_{j_0} is indeed an edge of the polytope. If j_0 is either 0 or n, this is certainly the case. Otherwise, we may choose integers j_1, j_2 such that $(0, j_1, v(f_{j_1}; \rho))$, $(0, j_0, v(f_{j_0}; \rho))$ and $(0, j_2, v(f_{j_2}; \rho))$ are distinct adjacent vertices of Λ_ρ, with $0\leqq j_1<j_0<j_2\leqq n$. Let P_1 be the plane determined by the lines $l_{j_0}(\rho)$, $l_{j_1}(\rho)$, and let P_2 be the plane determined by the lines $l_{j_0}(\rho)$, $l_{j_2}(\rho)$. By the concluding argument of Lemma 2, P_1 and P_2 are lower planes of support of the Newton polytope of $f(x, y)$. By choice of j_1 and j_2, they are distinct, and their intersection is the line $l_{j_0}(\rho)$. This completes the proof.

PROPOSITION 6. *If there is an edge of the Newton polytope of* $f(x, y)$ *with direction numbers* $(1, 0, -\rho)$, *then* $\rho\in\mathfrak{G}$ *and* ρ *is x-distinguished on* $V(f)$.

Note. It is not necessary to assume here that $\rho\in\mathfrak{G}$.

PROOF. We again write $f(x, y) = \sum_{j=1}^{n} f_j(x)y^j$; what we are required to show is that, if there is a polynomial $f_i(x)$ such that $f_i(x)$ has a zero on Γ_ρ and, moreover, that the side of Π_i of slope $-\rho$ is an edge of the Newton polytope $\mathbf{\Pi}$ of $f(x, y)$, then ρ is x-distinguished on $V(f)$. That is, we must show that the set $\mathfrak{L}_\rho = \{\lambda : -\lambda$ is the slope of a side of Λ_ξ, for some $\xi\in\Gamma_\rho\}$ is infinite. (We observe that $\rho\in\mathfrak{G}$, from the one-variable Newton polygon theory applied to $f_i(x)$.)

Case 1. *For some k,* $0\leqq k\leqq n$, f_k *has no zeros on* Γ_ρ. Let k_0 be the smallest such k. Then either $k_0=0$ or $k_0>0$.

(1a) Suppose $k_0=0$. Let i_0 be the smallest integer with the property that a side of Π_{i_0} of slope $-\rho$ is an edge of $\mathbf{\Pi}$. Then $(0, i_0, v(f_{i_0};\rho))$ is a vertex of Λ_ρ. Moreover, $i_0>0$, since f_0 has no zeros on Γ_ρ.

Let the vertices of Λ_ρ in the YZ-plane have Y-coordinates $0=\alpha_0<\alpha_1<\cdots<\alpha_t$, let $i_0=\alpha_\iota$. Then for all $\xi\in\Gamma_\rho$, the polygons Λ_ξ and Λ_ρ agree in vertices whose Y-coordinates are $\alpha_0, \alpha_1, \cdots, \alpha_{\iota-1}$.

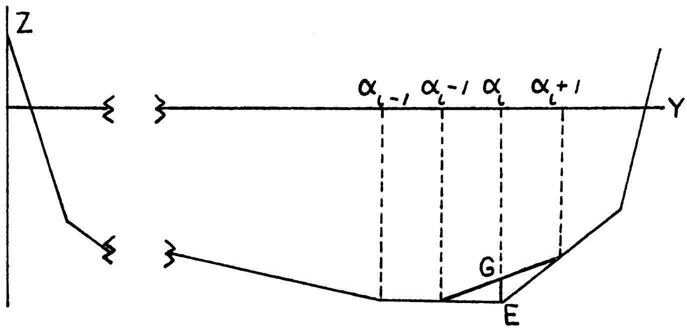

FIGURE 1. Newton polygon of Λ_ρ.

Consider the set $\mathfrak{Z}$ of Z-coordinates of points on the line $\overline{EG}$ in Figure 1 which are also in $\mathfrak{S}$.[1] Since $\mathfrak{S}$ is dense in R, the set $\mathfrak{Z}$ is infinite. But E has coordinates $(i_0, v(f_{i_0};\rho))$, whence from Proposition 1, we may choose, for each $r\in\mathfrak{Z}$, an element $\xi_r\in\Gamma_\rho$ such that $\operatorname{ord} f_{i_0}(\xi_r)=r$. Let Ξ be a set of representatives of $\mathfrak{Z}$ in Γ_ρ: if $\xi\in\Xi$ then $\operatorname{ord} f_{i_0}(\xi)\in\mathfrak{Z}$, and $\xi, \xi'\in\Xi$, $\xi\neq\xi'$ implies $\operatorname{ord} f_{i_0}(\xi)\neq\operatorname{ord} f_{i_0}(\xi')$. The definition of $\mathfrak{Z}$ guarantees that $(\alpha_\iota, \operatorname{ord} f_{\alpha_\iota}(\xi))$ is a vertex of Λ_ξ for any $\xi\in\Xi$.

For $\alpha_{\iota-1}\leqq\nu\leqq\alpha_\iota-1$, we define the sets $\Xi_\nu^{(1)}=\{\xi\in\Xi: (\nu, \operatorname{ord} f_\nu(\xi)), (i_0, \operatorname{ord} f_{i_0}(\xi))$ are vertices of a side of $\Lambda_\xi\}$. Then, since $\Xi=\bigcup_{\alpha_{\iota-1}\leq\nu\leq\alpha_\iota-1}\Xi_\nu^{(1)}$, we may choose ν_1 to be the largest integer with the property that $\nu_1<i_0$ and $\Xi_{\nu_1}^{(1)}$ is an infinite set. Let $\mathfrak{T}_1=\{\operatorname{ord} f_{\nu_1}(\xi): \xi\in\Xi_{\nu_1}^{(1)}\}$. If $\mathfrak{T}_1$ is finite, then the set $\{\operatorname{ord} f_{\nu_1}(\xi)-\operatorname{ord} f_{i_0}(\xi): \xi\in\Xi_{\nu_1}^{(1)}\}$ is infinite, whence so is $\mathfrak{L}_\rho$. Otherwise we define, for $\alpha_{\iota-1}\leqq\nu<\nu_1$, the sets $\Xi_\nu^{(2)}=\{\xi\in\Xi_{\nu_1}^{(1)}: (\nu, \operatorname{ord} f_\nu(\xi)), (\nu_1, \operatorname{ord} f_{\nu_1}(\xi))$ are vertices of a side of $\Lambda_\xi\}$, and choose ν_2 to be the largest integer with the property that $\nu_2<\nu_1$ and $\Xi_{\nu_2}^{(2)}$ is an infinite set. We then define $\mathfrak{T}_2=\{\operatorname{ord} f_{\nu_2}(\xi): \xi\in\Xi_{\nu_2}^{(2)}\}$. Proceeding in this manner, we define a sequence of integers $\nu_1>\nu_2>\cdots>\nu_m\geqq\alpha_{\iota-1}$, and a corresponding sequence of sets $\mathfrak{T}_1, \mathfrak{T}_2, \cdots, \mathfrak{T}_m$. If $\mathfrak{T}_i$ is finite for some i, then (1a) is proved. Otherwise, we may assume that m is such that $\nu_m=\alpha_{\iota-1}$; but then, $\mathfrak{T}_m=\{\operatorname{ord} f_{\alpha_{\iota-1}}(\xi): \xi\in\Xi_{\alpha_{\iota-1}}^{(m)}\}=\{v(f_{\alpha_{\iota-1}};\rho)\}$, which is certainly finite. Thus, case (1a) is proved.

[1] If $\alpha_\iota=n$, let $\mathfrak{Z}=\{\lambda\in\mathfrak{S}: \operatorname{ord}\lambda\geqq v(f_n;\rho)\}$.

(1b) Suppose $k_0>0$. Then f_0 has a zero on Γ_ρ. Hence, by Proposition 1, we may choose a sequence $\{\xi_h\}\subset\Gamma_\rho$ such that $\operatorname{ord} f_0(\xi_h)\to\infty$ as $h\to\infty$. If ρ is not x-distinguished on $V(f)$, we must then have $\operatorname{ord} f_1(\xi_h)\to\infty$ as $h\to\infty, \cdots, \operatorname{ord} f_{k_0-1}(\xi_h)\to\infty$ as $h\to\infty$. But then, for h sufficiently large, the point $(k_0, \operatorname{ord} f_{k_0}(\xi_h))=(k_0, v(f_{k_0};\rho))$ is on Λ_{ξ_h}.

For $0\leqq\nu<k_0$, we let $\mathcal{E}_\nu=\{h\in Z: (\nu, \operatorname{ord} f_\nu(\xi_h)), (k_0, \operatorname{ord} f_{k_0}(\xi_h))$ are vertices of a side of $\Lambda_{\xi_h}\}$. Then for some ν_0, the set $\mathcal{E}_{\nu_0}$ is infinite, whence the set of slopes

$$\left\{\frac{v(f_{k_0};\rho)-\operatorname{ord} f_{\nu_0}(\xi_h)}{k_0-\nu_0} : h\in\mathcal{E}_{\nu_0}\right\}$$

is infinite, whence ρ is x-distinguished on $V(f)$.

Case 2. *$f_j(x)$ has a zero on Γ_ρ for each j, $0\leqq j\leqq n$.* If there is an $\eta\in\Gamma_\rho$ such that $f_j(\eta)=0$ for each j, $0\leqq j\leqq n$, then certainly ρ is x-distinguished on $V(f)$. Therefore, we may assume that no root of one of the f_j is a root of all the f_j.

Let f_0 have the zeros $\eta_1, \eta_2, \cdots, \eta_w$ on Γ_ρ. Let j_0 be the smallest integer with the property that, for some i_0, $1\leqq i_0\leqq w$, $f_{j_0}(\eta_{i_0})\neq 0$. Choose a neighborhood N of η_{i_0} in $\mathfrak{K}$ such that $\{\operatorname{ord} f_{j_0}(\xi):\xi\in N\}$ is bounded. Since the valuation of $\mathfrak{K}$ is dense, we may choose a sequence $\{\xi_h\}\subset N$ such that $\operatorname{ord} f_k(\xi_h)\to\infty$, $h\to\infty$, for $k=0, 1, 2, \cdots, j_0-1$. But $\{\operatorname{ord} f_{j_0}(\xi_h): h=1, 2, \cdots\}$ is bounded; therefore, for h sufficiently large, the point $(j_0, \operatorname{ord} f_{j_0}(\xi_h))$ is on Λ_{ξ_h}.

For $0\leqq\nu<j_0$, let $\mathcal{E}_\nu=\{h\in Z: (\nu, \operatorname{ord} f_\nu(\xi_h)), (j_0, \operatorname{ord} f_{\nu_0}(\xi_h))$ are vertices of a side of $\Lambda_{\xi_h}\}$. Then for some ν_0, $\mathcal{E}_{\nu_0}$ is infinite; therefore, the set of slopes

$$\left\{\frac{\operatorname{ord} f_{j_0}(\xi_h)-\operatorname{ord} f_{\nu_0}(\xi_h)}{j_0-\nu_0} : h\in\mathcal{E}_{\nu_0}\right\}$$

is infinite, whence ρ is x-distinguished on $V(f)$.

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