ON THE INVERSE PROBLEM OF GALOIS THEORY
OF DIFFERENTIAL FIELDS

A. BIALYNICKI-BIRULA

0. One can ask what algebraic groups are isomorphic to groups of automorphism of strongly normal extensions of a fixed ordinary differential field (see [2]). The purpose of the note is to give a contribution in this direction. We shall prove the following theorem.

Theorem. Let $\mathcal{F}$ be an ordinary differential field with algebraically closed field of constants $C$ and suppose that $\mathcal{F}$ is of finite transcendence degree over $C$ but is different from $C$. Let $G$ be a connected nilpotent affine algebraic group defined over $C$. Then there exists a strongly normal extension $8$ of $\mathcal{F}$ such that the Galois group $G(\mathcal{E}/\mathcal{F})$ is isomorphic to $G(C)$.

1. All fields considered here are of characteristic 0. Let $F$ be a field, let $C$ be an algebraically closed subfield of $F$. Let $G$ be a connected algebraic group defined over $C$. $F(G)$ denotes the field of all rational functions on $G$ defined over $F$. If $g \in G$ then $F(g)$ denotes the field generated by $g$ over $F$. We shall say that a derivation of $F(G)$ commutes with $G^*(C)$ if it commutes with $g^*$, for every $g \in G(C)$, where $g^*$ denotes the automorphism of $F(G)$ induced by the left translation by $g$, i.e., $(g^*f)(x) = f(gx)$, for any $x \in G$. $\mathfrak{g}_F$ denotes the Lie algebra of all derivations of $F(G)$ that are zero on $F$ and which commute with $G^*(F)$. If $G_1$ is a normal subgroup of $G$ defined over $F$ then $F(G/G_1)$ is canonically isomorphic to a subfield of $F(G)$; we shall identify $F(G/G_1)$ and this subfield.

If $R$ is an integral domain then $(R)$ denotes the field of fractions of $R$. Every derivation $d$ of $R$ can be uniquely extended to a derivation of $R$ (the extended derivation will be also denoted by $d$). If $F_1, F_2$ are two fields containing $F$ as a subfield and if $d_1, d_2$ are derivations of $F_1, F_2$, respectively, such that $d_1|_F = d_2|_F$ and $d_1(F) \subset F$ then $d_1 \otimes d_2$ denotes the derivation of $F_1 \otimes_F F_2$ determined by $(d_1 \otimes d_2)(a \otimes b) = d_1(a) \otimes b + a \otimes d_2(b)$, for every $a \in F_1$ and $b \in F_2$.

$d_0$ denotes the zero derivation of a field (it will be always clear what field we have in mind). The underlying field of an ordinary differential field $\mathcal{F}$ will be denoted by $F$.

2. Lemma 1. If $d_1$ belongs to the center of $G_C$ then the derivation $d_1 \otimes d_0$ of $(C(G) \otimes F) (= F(G))$ commutes with every derivation $d$ of $F(G)$ such that $d(F) \subset F$ and $dg^* = g^*d$ for every $g \in G(C)$.

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Proof. Let \( d \) be as in the lemma. Then \( d - d_0 \otimes (d \mid F) \) is zero on \( F \) and commutes with \( G^*(F) \) and so \( d - d_0 \otimes (d \mid F) \in \mathfrak{S}_F \). But \( d_1 \otimes d_0 \) belongs to the center of \( \mathfrak{S}_F \) and commutes with \( d_0 \otimes (d \mid F) \). Thus \( d_1 \otimes d_0 \) commutes with \( d = d - d_0 \otimes (d \mid F) + d_0 \otimes (d \mid F) \).

**Lemma 2.** Let \( G_1 \) be a normal subgroup of \( G \) defined over \( C \) and let \( d^0 \) be a derivation of \( F(G/G_1) \) such that \( d^0(C) = 0 \) and \( d^0 \) commutes with any element from \( (G/G_1)^*(C) \). Then there exists an extension \( d' \) of \( d^0 \) to a derivation of \( F(G) \) that commutes with \( G^*(C) \).

Proof. Let \( g \) be a generic point of \( G \) over \( F(G) \). Extend \( d^0 \) to a derivation \( d_1 \) of \( F(G) \) and let \( d_2 \) be the extension of \( d_1 \) to a derivation of \( F(g)(G) \) which is trivial on \( C(g) \). Let \( V \) be a nonempty affine open subset of \( G \) defined over \( C \) and let \( C[x_1, \ldots, x_n] \) be the coordinate ring of \( V \) over \( C \). Then there exists \( h_0 \in V(C) \) such that \( d_1 x_1, \ldots, d_1 x_n \) are defined at \( h_0 \). Hence, if \( a \in F(g)(G) \) is defined at \( h_0 \) then \( d_2(a) \) is also defined at \( h_0 \). In particular, for any \( a \in F(G) \), \( d_2((gh_0^{-1})^*a) \) is defined at \( h_0 \) (since \( ((gh_0^{-1})^*a)(h_0) = a(g) \)). Let, for any \( a \in F(G) \), \( d'(a) \) be the element of \( F(G) \) such that \( d'(a)(g) = d_2((gh_0^{-1})^*a)(h_0) \). One can easily see that the definition of \( d' \) does not depend on \( g \). In particular, if \( g_1 \) is any point of \( G \) such that \( C(g_1) = C(g) \), then \( g_1 \) is generic for \( G \) over \( F \) and so \( d'(a)(g_1) = d_2((g_1h_0^{-1})^*a)(h_0) \). Hence, for any \( h \in G(C) \), \( (h^*d'(a))(g) = d'(a)(h g) = d_2((hg_0^{-1})^*a)(h_0) = d_2((gh_0^{-1})^*h^*a)(h_0) = d'(h^*a)(g) \), since \( C(h g) = C(g) \). Thus \( d_1 h^* = h^*d_1 \), i.e., \( d_1 \) commutes with \( G(C)^* \). Moreover, \( d' \) is a derivation of \( F(G) \). Indeed

\[
d'(a + b)(g) = d_2((gh_0^{-1})^*(a + b))(h_0) = d_2((gh_0^{-1})^*a)(h_0) + d_2((gh_0^{-1})^*b)(h_0)
= d'(a)(g) + d'(b)(g)
\]

and

\[
d'(ab)(g) = d_2((gh_0^{-1})^*ab)(h_0) = d_2((gh_0^{-1})^*a)(h_0) \cdot (gh_0^{-1})^*b(h_0) + (gh_0^{-1})^*a(h_0) \cdot d_2((gh_0^{-1})^*b)(h_0)
= d'(a)(g) \cdot b(g) + a(g) \cdot d'(b)(g).
\]

Finally, if \( a \in F(G/G_1) \) then

\[
d'(a)(g) = d_2((gh_0^{-1})^*a)(h_0) = d_0((gh_0^{-1})^*a)(h_0)
= (gh_0^{-1})^*d_0(a)(h_0) = d_0(a)(g),
\]

i.e., \( d' \) is an extension of \( d^0 \). This completes the proof of the lemma.

**Lemma 3.** Let \( G_1 \) be a connected central one-dimensional normal sub-
group of an affine connected algebraic group $G$, both defined over $F$. Let $d_1 \in \mathfrak{g}_F$ be a derivation in the direction of $G_1$. Then, for any $a \in F(G)$, $d_1(a) = 0$ if and only if $a \in F(G/G_1)$. Moreover, there exists an element $b \in F(G) - F(G/G_1)$ such that either $d_1(b) = c \cdot b$ or $d_1(b) = c$, where $c$ is an element from $F(G/G_1)$.

Proof. The first part of the lemma is well known. Let $b'$ be a regular function on $G$ such that $b' \in F(G) - F(G/G_1)$. Then $G^*_1(F) \cdot b'$ generates a finite-dimensional $F$-vector space. Since $G_1$ is one-dimensional and connected, hence we may assume that this space is either one-dimensional or two-dimensional with basis $b_0$, $b'$, where $b_0 \in F(G/G_1)$ and $g^*(b') = \alpha(g)b_0 + b'$, $\alpha \in F(G/G_1)$ and $\alpha(g) \neq 0$ if $g \neq$ identity $e$ of $G_1$. Then it follows from Lemma 7 [1] that in the first case $d_1(b') = cb'$, where $c$ is an element from $F(G/G_1)$ and we may take $b = b'$. In the second case (again by Lemma 7 [1]) $c_1d_1^2(b') + c_2d_1(b') + c_3b' = 0$, for some $c_1$, $c_2$, $c_3 \in F(G/G_1)$ which do not vanish simultaneously. Then $0 = g^*(c_1d_1^2(b') + c_2d_1(b') + c_3b') = c_1d_1^2(b') + c_2d_1(b') + c_3(\alpha(g)b_0 + b')$, for every $g \in G(G_1(F))$. Hence $c_3 = 0$ and $c_2 \neq 0$. If $c_2 \neq 0$, then $d_1(d_1(b')) = -c_2/c_1$, $d_1(b') \neq 0$, and we take $b = d_1(b')$. If $c_2 = 0$, then $d_1^2(b') = 0$. Hence $d_1(b') \in F(G/G_1)$, and we take $b = b'$.

Lemma 4. Let $\mathfrak{f}$ be an ordinary differential field with derivation $d$, let $C$ be the field of constants of $\mathfrak{f}$ and suppose that $SF$ is of finite transcendence degree over $C$. Let $\mathfrak{f}_1$ be a (differential) subfield of $\mathfrak{f}$ which is not contained in $C$ and let $c \in C$. Then there exist $a_1$, $a_2 \in \mathfrak{f}_1$, $a_1 \neq 0 \neq a_2$, such that there is no element $y \in \mathfrak{f} - C$ which satisfies either $dy = a_1 \cdot c$ or $dy = a_2cy$.

Proof. We may suppose that $\mathfrak{f}_1$ contains an element $x$ such that $dx = 1$ (let $x \in \mathfrak{f}_1 - C$; then $dx \neq 0$ and we may replace $d$ by $1/dx \cdot d$). If $dy_n = c/(x + n)$, $y_n \in \mathfrak{f} - C$, where $n$ is an integer, then, one can prove that the elements $y_n$ for different integers $i$, are algebraically independent over $C$. Similarly, if $dz_n = z^ncz_n$, then the elements $z_n$, for different integers $i$, are also algebraically independent over $C$. Hence $F$ contains only a finite number of elements $y_i$ and $z_n$. Thus, for some $n$, $y_n$, $z_n \in \mathfrak{f}$ and the lemma is proved.

3. Proof of the theorem. Let $d$ be the nonzero derivation of $\mathfrak{f}$. We shall show that one can extend $d$ to a derivation $d^*$ of $F(G)$ which commutes with $G^*(C)$ and has $C$ as the field of constants. Proof by induction on the dimension of $G$.

If $dim \ G = 0$, then this is trivial.

Suppose that the above is true for connected nilpotent affine groups of dimension $n$ and let $dim \ G = n + 1$. There exists a central connected
normal subgroup $G_1$ of $G$ defined over $C$ and of dimension 1. Then $G/G_1$ is an affine nilpotent connected group of dimension $n$. Hence there exists an extension $d^0$ of $d$ to a derivation of $F(G/G_1)$ such that $C$ is the field of constants of $d^0$ and $d^0$ commutes with $(G/G_1)^*(C)$. It follows from Lemma 2 that $d^0$ can be extended to a derivation $d'$ of $F(G)$ that commutes with $G^*(C)$. Let $d_1 \in \mathcal{G}_F$ be a derivation of $F(G)$ in the direction of $G_1$. Then the field of constants of $d_1$ is $F(G/G_1)$ and it follows from Lemma 1 that $d_1$ commutes with every derivation $ad'$, where $a \in F$. Therefore the set of all $b \in F(G)$, for which $d_1(b) = ad'(b)$, where $a$ is fixed, is a subfield $F_a$ of $F(G)$ closed under $d_1$ (and $ad'$). Indeed, it is easy to see that this is a field. Moreover, if $d_1(b) = ad'(b)$, then $d_1(d_1(b)) = d_1(ad'(b)) = ad'(d_1(b))$. $C$ is the field of constants of $d_1|F_a$ for $a \neq 0$, since the field of constants of $d_1$ is $F(G/G_1)$ and the field of constants of $ad'|F(G/G_1)$ is $C$. And we want to prove that $F_a = C$, for some $a \in F$. Let $a \in F$; consider the ordinary differential field $(F(G) \otimes F(G/G_1))$ together with the derivation $ad' \otimes d_0$ and the algebraic closure $(F(G) \otimes F(G/G_1))^*$ of $(F(G) \otimes F(G/G_1))$ with the unique extension $(ad' \otimes d_0)^*$ of $(ad' \otimes d_0)$. $F_a$ is linearly disjoint from $F(G/G_1)$ over $C$ since $F(G/G_1)$ is the field of constants of $d_1$ and $C$ is the field of constants of $d_1|F_a$ (see Proposition 1 in [3] or Lemma 1 in [1]). Hence there exists a subfield of $F(G)$ with $d_1$ which is canonically isomorphic to $(F_a \otimes F(G/G_1))$ with $(ad' \otimes d_0) < F(G)$ is an algebraic extension of the subfield unless $F_a = C$ and this isomorphism maps $b$ onto $1 \otimes b$, for every $b \in F(G/G_1)$. Therefore $F_a \neq C$ implies that there exists an isomorphism $\alpha_a$ of $F(G)$ with $d_1$ into $(F(G) \otimes F(G/G_1))^*$ with $(ad' \otimes d_0)^*$ such that $\alpha_a(b) = 1 \otimes b$, for every $b \in F(G/G_1)$. It follows from Lemma 3 that there exist elements $c \in F(G/G_1)$ and $y \in F(G) - F(G/G_1)$ such that either $d_1y = c$ or $d_1y = cy$. Therefore, for every $a \in F$, $a \neq 0$ for which $F_a \neq C$, we have that either $(ad' \otimes d_0)^*\alpha_a(y) = 1 \otimes c$ or $(ad' \otimes d_0)^*\alpha_a(y) = 1 \otimes c/a \otimes 1$ or $(ad' \otimes d_0)^*\alpha_a(y) = 1 \otimes c/a \otimes 1 \otimes \alpha_a(y)$. But it follows from Lemma 4 that there exist $a_1$, $a_2 \in F$ such that neither $(d' \otimes d_0)^*z = 1 \otimes c/a_1 \otimes 1$ nor $(d' \otimes d_0)^*z = 1 \otimes c/a_2 \otimes 1 \otimes z$ has a solution $z$ in $(F(G) \otimes F(G/G_1))^*$. Then $a_1 \neq 0 \neq a_2$ and either $F_{a_1} = C$ or $F_{a_2} = C$. If $F_a = C$, then $a \neq 0$ and the field of constants of $d^* = (1/a)d_1 - d'$ is $C$. Moreover, $d^*$ commutes with $G^*(C)$. Thus we have proved by induction that there exists an extension $d^*$ of $d$ that commutes with $G^*(C)$ and has $C$ as the field of constants.

Now if $d^*$ is such a derivation then $F(G)$ with $d^*$ is a strongly normal extension of $\mathcal{G}$ and $G(C)$ is the Galois group of the extension (see Proposition 1 and Theorem 1 in [1]).
ON A REALIZATION OF A COMPLEMENTED ALGEBRA

PARFENY P. SAWOROTNOW

In this note we intend to show that each simple complemented algebra is isomorphic to an algebra described in the example below (as in [6] we use the term "simple" to mean "simple and semisimple"). This paper can be considered as a continuation of [5] and [6].

In the example below (and in the proof of the theorem after it) we use terms "summable" and "integrable" in the sense defined in Chapter III of [3].

Example. Let \((S, \mu)\) be a measure space. Let \(K(s)\) be a real-valued function defined on \(S\) and having the following properties:

(i) \(K(s)\) is finite almost everywhere,

(ii) there exists a positive number \(a\) such that \(aK(s)\) for each \(s \in S\),

(iii) the restriction of \(K(s)\) to any summable subset of \(S\) is integrable (in particular \(K(s)\) may be integrable).

Let \(A\) be the set of all complex-valued members \(x\) of \(L^2(S \times S, \mu \times \mu)\) such that \(\int \int |x(t, s)|^2 K(s) \, dt \, ds\) is finite. Then \(A\) is a complemented algebra in the scalar product \((x, y) = \int \int x(t, s) y(t, s) K(s) \, dt \, ds\) and the multiplication \((xy)(t, s) = \int x(t, r) y(r, s) \, dr\) (we consider pointwise addition and pointwise multiplication with a scalar). If \(K(s)\) is bounded above then \(A\) is well complemented. Condition (ii) implies continuity of the multiplication (in both factors simultaneously); if \(a = 1\) then \(\|xy\| \leq \|x\| \|y\|\).

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References