REFERENCES

1. C. Berge, *The theory of graphs and its applications*, Wiley, New York, 1962.
2. R. Rado, *Axiomatic treatment of rank in infinite sets*, Canad. J. Math. 1 (1949), 337–343.

MARATHON OIL COMPANY, LITTLETON, COLORADO

CONCERNING CONTINUOUS IMAGES OF COMPACT ORDERED SPACES

L. B. TREYBIG[1]

It is the purpose of this paper to prove that if each of X and Y is a compact Hausdorff space containing infinitely many points, and $X\times Y$ is the continuous image of a compact ordered space L, then both X and Y are metrizable.[2] The preceding theorem is a generalization of a theorem [1] by Mardešić and Papić, who assume that X, Y, and L are also connected. Young, in [3], shows that the Cartesian product of a "long" interval and a real interval is not the continuous image of any compact ordered space.

In this paper, the word compact is used in the "finite cover" sense. The phrase "ordered space" means a totally ordered topological space with the order topology. A subset M of a topological space is said to be heriditarily separable provided each subset of M is separable. If a and b are points of an ordered space L and $a<b$, then $[a, b]$ ((a, b)) will denote the set of all points x of L such that $a\leqq x\leqq b$ ($a<x<b$), provided there is one; also, $[a, b]$ will be used even in the case where $a=b$. A subset M of an ordered space L is convex provided that if $a\in M$, $b\in M$, and $a<b$, then $[a, b]\subset M$. If M is a subset of an ordered space L, then $G(M)$ will denote the set of all ordered pairs (a, b) such that (1) $a\in M$, $b\in M$, and $a<b$, and (2) $\{a, b\} = M\cdot[a, b]$, provided there is one.

LEMMA 0. *If M is a compact subset of the ordered space L, then the relative topology of L on M is the same as the order topology on M.*

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LEMMA 1. *If M is a nondegenerate, totally disconnected, compact subset of an ordered space L, then M is metrizable if and only if $G(M)$ is countable.*

PROOF. Suppose M is metrizable. Since a compact Hausdorff space is metric if and only if it satisfies the second axiom of countability, there is a countable sequence $I_1, I_2, \cdots$ such that (1) for each n, I_n is a convex open subset of L, and (2) $I_1 \cdot M, I_2 \cdot M, \cdots$ is a countable basis for M. There exists a transformation T from $G(M)$ into the ordered pairs of positive integers such that if $(a, b) \in G(M)$ and $T((a, b)) = (p, q)$, then $I_p \cdot \{a, b\} = a$ and $I_q \cdot \{a, b\} = b$. T is easily seen to be a one-to-one transformation, so $G(M)$ is evidently countable.

Suppose $G(M)$ is countable. Let the elements of $G(M)$ be labeled $(a_1, b_1), (a_2, b_2), \cdots$. Let H denote a collection such that $h \in H$ if and only if (1) there is a positive integer i such that h is the set of all points of M which precede b_i, or h is the set of all points of M which follow a_i; or (2) there exist integers i and j such that $h = M \cdot [b_i, a_j]$. H is a countable basis for M, so M is metrizable.

LEMMA 2. *If M is a separable subset of the ordered space L, then M is hereditarily separable.*

PROOF. Suppose H is a subset of M. There is a countable set $P_1, P_2, \cdots$ dense in M such that if $P \in M$, then for some integer pair (i, j), $P_i \leqq P \leqq P_j$. For each integer pair (i, j) such that $P_i \leqq P_j$ and $[P_i, P_j] \cdot H$ exists, let H_{ij} denote a countable subset of $[P_i, P_j] \cdot H$ such that if $P \in [P_i, P_j] \cdot H$, then there exists R in H_{ij} and S in H_{ij} such that $R \leqq P \leqq S$. $\sum H_{ij}$ is easily seen to be a countable set dense in H.

LEMMA 3. *If M is a nonconnected, separable, compact subset of the ordered space L, then M is metrizable if and only if $G(M)$ is countable.*

LEMMA 4. *If the continuous function f_1 maps the compact ordered space K_1 onto the Hausdorff space S, then there is a compact ordered space K_2 and a continuous function f_2 mapping K_2 onto S such that (1) if K is a closed proper subset of K_2, then $f_2(K) \neq S$, and (2) if x and y are elements of K_2 such that $f_2(x) = f_2(y)$, there is an element z of K_2 between x and y such that $f_2(z) \neq f_2(x)$.*

PROOF. Let H denote the set of closed subsets m of K such that $f_1(m) = S$. Define a partial order $\leqq$ on H by saying $m_1 \leqq m_2$ if and only if $m_1 \subset m_2$. It is easily verified that each chain has a lower bound, so Zorn's lemma applies here, and H has a minimal element K.

For each point x of K let K_x denote the union of all the subsets k of K such that (1) $x \in k$, (2) k is convex relative to K, and (3) if $y \in k$, then $f_1(y) = f_1(x)$. Each K_x is closed, and if x and y are elements of K, then either $K_x = K_y$ or $K_x \subset K - K_y$. Let K_2 denote the set of all K_x for $x \in K$, and suppose U is open in K_2 if and only if U^* is open in K.[3] Suppose that K_2 is given the natural order induced by the order on K, and that f_2, which maps K_2 onto S, is defined by $f_2(K_x) = f_1(x)$. The space K_2 and the function f_2 satisfy the conclusion of the lemma.

LEMMA 5. *If the continuous function f maps the compact metric space R onto the Hausdorff space S, then S is metrizable.*

PROOF FOR THE CASE WHERE S IS NONDEGENERATE. Let $R_1, R_2, \cdots$ denote a countable basis for R. Let T denote a collection such that $U \in T$ if and only if there exists two points s_1 and s_2 of S and a finite integer sequence $j_1, j_2, \cdots, j_n$ such that $f^{-1}(s_1) \subset \sum R_{j_i}$, $f^{-1}(s_2) \subset R - \sum R_{j_i}$, and $U = S - f(R - \sum R_{j_i})$. The collection T is a countable basis for the compact Hausdorff space S, so S is metrizable.

LEMMA 6. *If the continuous function f maps the compact ordered space K onto the infinite, compact Hausdorff space S, then there exists a sequence $x_0, x_1, \cdots$ of distinct elements of S such that $x_1, x_2, \cdots$ converges to x_0.*

PROOF. Let $y_1, y_2, \cdots$ denote a sequence of distinct elements of S, and for each n, let z_n denote an element of $f^{-1}(y_n)$. There is an increasing sequence of integers $n_1, n_2, \cdots$ such that $z_{n_1}, z_{n_2}, \cdots$ is monotone, and since K is compact, there is a point z such that the latter sequence converges to z. There is a subsequence $j_1, j_2, \cdots$ of $n_1, n_2, \cdots$ such that $f(z_{j_i}) \neq f(z)$, $i = 1, 2, \cdots$. The sequence $x_0, x_1, \cdots$ defined by $x_0 = f(z)$ and $x_i = f(z_{j_i})$, $i \geqq 1$, satisfies the conclusion of the lemma.

PROOF OF THEOREM. Suppose that X is not metrizable. Let u denote an element of X, and let g map L continuously onto $X \times Y$. Since $\{u\} \times Y$ is the continuous image of a compact ordered space, an application of Lemma 6 yields an infinite sequence of distinct points (u, b), (u, b_1), (u, b_2), $\cdots$, all lying in $\{u\} \times Y$, such that (u, b_1), (u, b_2), $\cdots$ converges to (u, b). The space $Z = \{b, b_1, b_2, \cdots\}$ with the relative topology of Y is an infinite compact Hausdorff space, and the space $X \times Z$ is the continuous image of a compact ordered space, so there exists a compact ordered space K and a continuous function f mapping K onto $X \times Z$ such that the conclusions of Lemma 4 hold.

[3] If U is a collection of point sets, then U^* denotes the sum of the sets of the collection U.

For each positive integer n, let H_n denote a partition of $X \times Z$ into the following $n+1$ open and closed sets: $X \times \{b_1\}$, $X \times \{b_2\}, \cdots,$ $X \times \{b_n\}$, $X \times \{b, b_{n+1}, b_{n+2}, \cdots \}$. For each n, let K_n denote the set of all $f^{-1}(h)$ for $h \in H_n$, and let I_n denote a partition of K into convex open and closed sets such that if $I \in I_n$, there is an element k of K_n such that $I \subset k$. Since K is compact, each I_n is a finite collection. Let C denote a point set to which a point P belongs if and only if there is an integer n and an element I of I_n which intersects $f^{-1}(X \times \{b\})$ such that P is either the right-most point of this intersection or the left-most. C is a countable set which will be shown to be dense in $f^{-1}(X \times \{b\})$.

Suppose the set $f^{-1}(X \times \{b\})$ contains an open set U. Let P denote an arbitrary point of $f^{-1}(X \times \{b\})$ and suppose $f(P) = (x, b)$. For each n let Q_n denote an element of $f^{-1}(x, b_n)$. Some subsequence of the Q_i's converges to a point Q in $K - U$, and the continuity of f implies that $f(Q) = (x, b)$. Therefore, $f(K - U) = X \times Z$, which is a contradiction. Now suppose that $P \in f^{-1}(X \times \{b\})$ and $R < P < S$. There is a positive integer n and a point Q of $f^{-1}(X \times \{b_n\})$ in (R, S). Suppose $P < Q < S$. There is an element I of I_n containing P, but not Q, and the right-most point T of $I \cdot f^{-1}(X \times \{b\})$ is an element of C satisfying $P \leqq T < S$. This case clearly shows why C is dense in $f^{-1}(X \times \{b\})$.

The separability of $f^{-1}(X \times \{b\})$ implies that $X \times \{b\}$ is separable, and consequently, that $X \times Z$ is separable. Let $\{R_1, R_2, \cdots \}$ denote a countable set dense in $X \times Z$, and for each n let P_n denote an element of $f^{-1}(R_n)$. The set $K' = \text{cl}(\sum P_i)$ is a closed subset of K such that $f(K') = X \times Z$, so $K' = K$ and K is separable.

It will now be shown that $X \times Z$ satisfies the first axiom of countability.[4] Let P denote an arbitrary point of $X \times Z$. Since $f^{-1}(P)$ is compact and $K - f^{-1}(P)$ is separable, it follows by an easy argument that there is a countable set $\{Q_1, Q_2, \cdots \}$ dense in $K - f^{-1}(P)$ such that if $x \in f^{-1}(P)$ and $y \in K - f^{-1}(P)$, there is a Q_i such that $x < Q_i \leqq y$ or $y \leqq Q_i < x$. For each positive integer n, let V_n denote a collection to which v belongs if and only if there is a point z of $f^{-1}(P)$ such that v is the maximal convex subset of K which contains z and does not intersect $\sum_1^n Q_i$. Since, for each n, V_n^* is an open subset of K containing $f^{-1}(P)$, it follows that the set $T_n = X \times Z - f(K - V_n^*)$ is an open subset of $X \times Z$ containing P. Suppose Q is an arbitrary point of $X \times Z$ distinct from P, that $z \in f^{-1}(Q)$, and also, for example, that z_1 is the last point of $f^{-1}(P)$ which precedes z and z_2 is the first point of $f^{-1}(P)$ which follows z. There exist an integer j_1 and an integer j_2

[4] It also may be shown from [2] that $X \times Z$ satisfies the first axiom of countability.

such that $z_1 < Q_{j_1} \leqq z \leqq Q_{j_2} < z_2$. The set V_j^*, where $j = \max(j_1, j_2)$, does not contain z, so the set T_j does not contain Q. Therefore, $T_1, T_2, \cdots$ is a countable sequence of open sets having only P in common.

The set $f^{-1}(X \times \{b_1\})$ is not metrizable, since that would imply that X is metrizable. Since Lemma 2 implies that $f^{-1}(X \times \{b_1\})$ is separable, Lemma 3 implies that $G_1 = G(f^{-1}(X \times \{b_1\}))$ is uncountable. There does not exist an uncountable subcollection U_1 of G_1 such that if $(x, y) \in U_1$, then $f(x) = f(y)$; for if there does, the conditions on f imply that for $(x, y) \in U_1$, there is a P_i such that $x < P_i < y$, which is a contradiction. Suppose there is an uncountable subcollection U_2 of G_1 and a point x of X such that if $(z, w) \in U_2$, then $f(z) = (x, b_1)$ or $f(w) = (x, b_1)$. There is an uncountable subcollection U_3 of U_2 such that if $(z, w) \in U_3$ and $f(z) = (x, b_1)$, then $f(w) \neq (x, b_1)$. The fact that $X \times Z$ has a countable basis at (x, b_1) implies that there is an open set U containing (x, b_1) and an uncountable subcollection U_4 of U_3 such that if $(z, w) \in U_4$ and $f(z) \in U$, then $f(w) \in (X \times Z) - U$. There is a point t of K such that each open set containing t contains uncountably many elements (z, w) of U_4. The continuity of f would imply that $f(t) = (x, b_1)$ and that $f(t) \in (X \times Z) - U$, which is a contradiction.

Let C denote the collection of all subsets M of G_1 such that if (p, q) and (p', q') are elements of M then $f(p)$, $f(q)$, $f(p')$, and $f(q')$ are four distinct points. C is partially ordered by inclusion, and each chain has an upper bound, so Zorn's lemma implies the existence of a maximal element W. Suppose W is countable. Let D denote the set of all elements (p, q) of G_1 such that there is an element (p', q') of W such that $f(p) = f(p')$ or $f(q')$, or $f(q) = f(p')$ or $f(q')$. D is countable, so there is an element (p, q) of $G_1 - D$ such that $f(p) \neq f(q)$. However, $W + \{(p, q)\}$ is an element of C containing W, so W is not maximal. This is a contradiction, so W is uncountable.

It will now be shown that if x_1 and x_2 are points in X, then there is a positive integer N such that if $n > N$, $z_1 \in f^{-1}(x_1, b_n)$, and $z_2 \in f^{-1}(x_2, b_n)$, then there is a point of K between z_1 and z_2. On the contrary, suppose there exist points x_1 and x_2 of X and an increasing sequence of integers $n_1, n_2, \cdots$ such that for each i there exist points z_i and w_i of $f^{-1}(x_1, b_{n_i})$ and $f^{-1}(x_2, b_{n_i})$, respectively, such that no point of K lies between z_i and w_i. There is a point z of K such that each open set about z contains, for infinitely many integers i, both z_i and w_i. But the continuity of f would imply that $f(z) = (x_1, b)$ and also that $f(z) = (x_2, b)$, which is a contradiction.

Let V denote the set of all ordered pairs (x, y) such that there is an element (z, w) of W such that $f(z) = (x, b_1)$ and $f(w) = (y, b_1)$. There is a positive integer N and an uncountable subcollection V_1 of V such that if $(x, y) \in V_1$, $z \in f^{-1}(x, b_N)$, and $w \in f^{-1}(y, b_N)$, then there is some point of K between z and w. Let T_1 denote a set to which t belongs if and only if there exist integers i and j such that t is maximal with respect to the property of being a convex subset of K which contains neither P_i nor P_j. Let T_2 denote a collection to which t belongs if and only if $t\in T_1$ or t is the union of a finite number of elements of T_1. The collection T_2 is countable and has the property that if $(x, y)\in V_1$ then there exist elements t_1 and t_2 of T_2 such that $f^{-1}(x, b_N)\subset t_1\subset K-f^{-1}(y, b_N)$ and $f^{-1}(y, b_N)\subset t_2\subset K-f^{-1}(x, b_N)$. This is easily seen, because for each z in $f^{-1}(x, b_N)$, for example, there is an element t_z of T_1 which contains z and does not intersect $f^{-1}(y, b_N)$, and $f^{-1}(x, b_N)$ is covered by a finite number of the t_z's.

Let S_1 denote a collection to which an element s belongs if and only if there is an element t of T_2 such that $(s\times\{b_N\})=X\times\{b_N\}-f(K-t)\cdot(X\times\{b_N\})$. S_1 is a countable collection of open subsets of X such that if $(x, y)\in V_1$, there exist elements s_1 and s_2 of S_1 such that $x\in s_1\subset X-\{y\}$ and $y\in s_2\subset X-\{x\}$. Since S_1 is countable and V_1 is uncountable, there is an element s of S_1 and an uncountable subcollection V_2 of V_1 such that if $(x, y)\in V_2$, then $x\in s\subset X-\{y\}$. Since f is continuous and $s\times\{b_1\}$ is open in $X\times Z$, it follows that $f^{-1}(s\times\{b_1\})$ is open in K.

Let W_1 denote the collection of all elements (c, d) of W such that there is an element (x, y) of V_2 such that $(f(c); f(d))=(x, b_1; y, b_1)$. If $(c, d)\in W_1$, $c\in f^{-1}(s\times\{b_1\})$ and $d\in K-f^{-1}(s\times\{b_1\})$. For each pair (c, d) of W_1 let $U(c)$ denote a convex open subset of K such that $c\in U(c)$ and $U(c)\subset f^{-1}(s\times\{b_1\})$. The set of all $U(c)$'s is uncountable and no two of them intersect, so K is not separable. This yields a contradiction, so X is metrizable.

One interesting application of the preceding theorem is the following

THEOREM. *If a space X is the continuous image of a compact ordered space and can be expressed as an infinite product* ($\prod X_i$), *where each X_i is a nondegenerate compact Hausdorff space, then* (1) *the product is a countable product, and* (2) *each X_i is metrizable.*

REFERENCES

1. S. Mardešić and P. Papić, *Continuous images of ordered continua*, Glasnik Mat.-Fiz. Astronom. **15** (1960), 171–178.

2. ———, *Diadic bicompacta and continuous mappings of ordered bicompacta*, Dokl. Akad. Nauk SSSR **143** (1962), 529–531. (Russian)

3. G. S. Young, *Representations of Banach spaces*, Proc. Amer. Math. Soc. **13** (1962), 667–668.

TULANE UNIVERSITY