A NOTE ON COMPACT SEMIRINGS

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By a topological semiring we mean a Hausdorff space S together with two continuous associative operations on S such that one (called multiplication) distributes across the other (called addition). That is, we insist that $x(y+z)=xy+xz$ and $(x+y)z=xz+yz$ for all x, y, and z in S. Note that, in contrast to the purely algebraic situation, we do not postulate the existence of an additive identity which is a multiplicative zero.

In this note we point out a rather weak multiplicative condition under which each additive subgroup of a compact semiring is totally disconnected. We also give several corollaries and examples.

Following the notation current in topological semigroups we let $H[+](e)$ represent the maximal additive subgroup containing an additive idempotent e. Similarly $H[\cdot](f)$ will denote the maximal multiplicative group of a multiplicative idempotent f. The minimal closed additive or multiplicative semigroup containing x is denoted by $\Gamma[+](x)$ or $\Gamma[\cdot](x)$ respectively. By $E[+]$ or $E[\cdot]$ we mean the collection of additive or multiplicative idempotents. Finally A^* represents the topological closure of A. For references on the properties of these sets the reader may see [1].

THEOREM. *If S is a compact semiring such that $E[\cdot]S\cup SE[\cdot]=S$ then each additive subgroup of S is totally disconnected.*

PROOF. Let S be a compact semiring such that $E[\cdot]S\cup SE[\cdot]=S$ and let G be an additive subgroup with identity e. Suppose G is not totally disconnected. Then neither is $H[+](e)$ since $G\subset H[+](e)$. That is, C, the component of e in $H[+](e)$, is nontrivial. Now C is a compact connected nontrivial group. It is well known [2, pp. 175, 190, and 192] that C must contain a nontrivial additive one parameter group T. Pick t different from e in T. Recall that $t\in fS$ or Sf for some f in $E[\cdot]$. Suppose $t\in fS$. Clearly fS is a compact subsemiring for which f is a multiplicative left identity. Thus $ft=t$ so fT is nontrivial and of course, fT is connected. Therefore fS contains a connected nontrivial group. Similarly if $t\in Sf$ then Sf is a compact subsemiring with right identity containing a connected nontrivial subgroup. Thus, without loss of generality, we may assume S has a left or a right identity 1.

Suppose 1 is a left identity. We identify each positive integer n with

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the n-fold sum of 1. Thus, if $x \in S$, we may regard nx as a product as well as a sum in S. Now, for each positive integer n, we have $nT = T$ so $(nT)^* = T^*$. This and the compactness of S gives us $nT^* = T^*$. From a theorem of A. D. Wallace [3], we see that $xT^* = T^*$ for each x in $\Gamma[+](1)$. But, $\Gamma[+](1)$ contains an additive idempotent g. Thus $gT^* = T^*$. On the other hand the additive idempotents $E[+]$ form a multiplicative ideal, so $T^* \subset E[+]$. Thus the additive group T must consist of a single element. This is a contradiction. Since a similar argument applies in case S contains a right identity, the theorem is proved.

By a *clan* we mean a compact connected semigroup with identity. Furthermore, we say a space S is *acyclic* provided $H^n(S) = 0$ for each positive integer n where $H^n(S)$ represents the nth Čech cohomology group of S.

COROLLARY 1. *If S is a compact semiring such that $E[\cdot]S \cup SE[\cdot] = S$ then every additive subclan is acyclic.*

PROOF. Let T be an additive subclan of S. Let e be an additive idempotent of the minimal additive ideal of T. Then it is known that $e+T+e$ is a group and $H^n(T) = H^n(e+T+e)$ for each positive integer n [4]. Now, by the theorem, $e+T+e$ is totally disconnected. But $e+T+e$, being a continuous image of T, is connected. Thus $e+T+e$ is a single point and $H^n(T) = 0$ for $n > 0$.

A semigroup S is said to be *normal* if $Sx = xS$ for each x in S.

COROLLARY 2. *Let S be a compact semiring such that $SE[\cdot] \cup E[\cdot]S = S$. Let T be an additive subsemigroup of S.*

(i) *If T is a continuum then the minimal ideal of T is idempotent.*

(ii) *If also T is normal and $(E[+] \cap T) + T = T$ then each closed ideal of T is acyclic.*

(iii) *If also T is metric then T is arcwise connected.*

(iv) *If also $T = S$ then there is an element k in S such that $k+S = S+k = k = k^2$.*

PROOF. Let S be a compact semiring such that $SE[\cdot] \cup E[\cdot]S = S$ and T be an additive semigroup of S. Suppose T is a continuum. It is well known [5] that the minimal ideal of T is the union of groups each of which is of the form $e+T+e$ where $e \in E[+] \cap T$. Now $e+T+e$, being a continuous image of T, is a continuum. But each additive subgroup of S must be totally disconnected. Consequently, $e+T+e$ is a single point and we have shown that the minimal ideal of T consists entirely of idempotents.

Suppose also that T is normal and $(E[+] \cap T) + T = T$. The normality of T implies that its minimal ideal is a group. Since the minimal ideal of T is also idempotent it must be a single point, say k. Now Corollary 3 of [6] gives us that each closed ideal of T is acyclic and part (ii) is proved.

Assume in addition that T is metric and select an additive idempotent f of T. Clearly $f+T$ is a compact connected additive subsemigroup of T. Furthermore $k\in f+T$ and, because $f+T=T+f$, f is an additive identity for $f+T$. Pick x in $f+T$. We have $x+(f+T)=(x+f)+T=x+T=T+x=T+(f+x)=(T+f)+x=(f+T)+x$. Thus $f+T$ is additively normal. Also, by the theorem, each additive subgroup of $f+T$ is totally disconnected. Now R. P. Hunter has shown that each such semigroup contains an arc (indeed, an I-semigroup) from the zero to the identity [7, Theorem 1]. Since $f+T$ is metric it is arcwise connected. But $T=\bigcup\{f+T \mid f\in E[+]\cap T\}$ and $k\in f+T$ for each f in $E[+]\cap T$. Therefore T is arcwise connected.

Suppose $T=S$ so that $S+k=k+S=k$. Now k is in either $E[\cdot]S$ or $SE[\cdot]$. In the former case there is a g in $E[\cdot]$ such that $gk=k$. But because k is an additive zero we have $k=g+k$ and $k=k+k^2$. Thus $k=k+k^2=(g+k)k=k^2$. In case $k\in SE[\cdot]$ a similar argument gives us that $k=k^2$ and the corollary is proved.

Recall that if the minimal ideal K of a compact semigroup S is idempotent then its structure is completely known [8]. That is, K must be of the form $A\times B$ where multiplication is defined by $(a, b)\cdot(a', b')=(a, b')$ for any a and a' in A and b and b' in B.

If S is a semiring, let $H[+]$ denote the union of all the additive subgroups of S.

Corollary 3. *Let S be an additively commutative semiring such that $SE[\cdot]\cup E[\cdot]S=S$. If S is a metric continuum and $e\in E[+]$ then $E[+]$, $E[+]+S$, $e+S$, $H[+]$ and $e+H[+]$ are arcwise connected.*

Proof. Clearly $E[+]$ is a closed additive subsemigroup and a multiplicative ideal. Now, since S is connected and $SE[\cdot]\cup E[\cdot]S=S$, the multiplicative ideals of S are connected and $E[+]$ is a continuum. From this it follows that $E[+]+S$ is also a continuum. Furthermore, $e+S$ being a continuous image of S, is a continuum. Applying the third part of Corollary 2, we have that $E[+]$, $E[+]+S$, and $e+S$ are arcwise connected. Now suppose $x\in S$ and G is an additive subgroup of S. It follows from the distributive property that xG and Gx are additive subgroups of S. Thus $xH[+]\cup H[+]x\subset H[+]$. That is, $H[+]$ is a multiplicative ideal of S. It is well known [1] that $H[+]$ is a closed additive subsemigroup of S. Thus Corollary 2 gives us that $H[+]$ is arcwise connected.

Notice that for $E[+]$ to be a continuum it is only necessary that S be a continuum and $E[\cdot]S\cup SE[\cdot]=S$. In case $E[+]$ is a single point k we have $k+S=S+k=kS=Sk=k$. To see this, recall that $E[+]$ is a multiplicative ideal and hence must contain the minimal such. On the other hand, the minimal additive ideal of S consists of idempotents and therefore must be k.

A semigroup is said to be *simple* if it contains no proper ideals.

COROLLARY 4. *If S is a compact connected additively simple semiring then each multiplicative idempotent of S is an additive idempotent of S.*

PROOF. Let e be a multiplicative idempotent of S. Then eS is a compact connected subsemiring for which e is a multiplicative left identity. Now eS is additively simple since it is additively a homomorphic image of S. Thus the first part of Corollary 2 gives us that eS is additively idempotent. But $e\in eS$ so e is an additive idempotent and the corollary is proved.

EXAMPLE. Let A be the field of integers mod 3 with the discrete topology and B be the interval $[0, 1]$. For x and y in B, define $x+y=xy=\min\{x, y\}$. Note that B is a semiring so that $A\times B$ becomes a semiring under coordinate-wise addition and multiplication. Define the equivalence relation α on $A\times B$ by: $(a, j)\alpha(a', j')$ if (1) $a=a'$ and $j=j'$ or (2) $j=j'=0$. Clearly α is a closed congruence. Thus $(A\times B)/\alpha$ is a compact connected semiring with multiplicative identity. The maximal additive subgroups of $(A\times B)/\alpha$ are of the form $(A\times\{b\})/\alpha$ and of course totally disconnected.

On the other hand let C be the circle group written additively and given the multiplication $xy=0$ for all x and y in C. According to the theorem, C can not be imbedded in a compact semiring with multiplicative identity (even if the identity is isolated).

QUESTION. Regarding the proof of the third part of Corollary 2, it is easily seen that $e+T$ is not only arcwise connected but also contractable. Indeed $(e+T)\cup(f+T)$ is contractable for e and f in $E[+]\cap T$. The referee has pointed this out and raised the question: Is T contractable?

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ON SOME GEOMETRIC INEQUALITIES

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1. Let C be a closed curve of class C^2 in Euclidean n-space E_n. We write the equation of C as $\mathbf{x}=\mathbf{x}(s)$, $0\leqq s\leqq L(C)$, where s denotes arc length and $L(C)$ is the length of C. Denoting differentiation with respect to s by a dot, we define the *total curvature* of C as

$$K(C) = \int_C |\ddot{\mathbf{x}}|\, ds. \tag{1}$$

It is proved in [1] that if C is constrained to lie in a ball of radius r, then

$$L(C) \leqq rK(C). \tag{2}$$

This result is a slight sharpening of an inequality of I. Fáry [2]. The proof given in [1] depends on an integralgeometric lemma for the 2-dimensional case, together with a reduction of the n-dimensional to the 2-dimensional case by developing the curve into a plane. The proof yields no information about curves for which equality occurs in (2).

In §2 we give a simple, direct proof of (2) and characterize those curves for which equality holds. We also obtain a sharpening of an inequality of Rešetnjak [3]. A generalization to surfaces is considered in §3.

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