algebraic theorem can be proved in its full generality only by resorting to the geometric lemma. This is in direct contrast to the situation where one can prove the geometric theorem that a finite Desarguesian plane satisfies Pappus's theorem only by using the algebraic fact that a finite division ring is commutative.

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Institute for Defense Analyses

PARTITIONS WITH EQUAL PRODUCTS

JOHN B. KELLY

T. S. Motzkin has conjectured (oral communication) that every sufficiently large positive integer can be partitioned into three positive integral parts in two different ways so that the products of the integers in the two partitions are equal (e.g., $13=1+6+6=2+2+9$; $1\cdot 6\cdot 6=2\cdot 2\cdot 9=36$). In this note we prove a generalization of this conjecture.

Theorem. *Let k be an integer $\geqq 3$. There exists an integer $N(k)$ such that every integer $n \geqq N(k)$ can be partitioned into k parts in $k-1$ different ways:*

$$(1)\qquad \begin{aligned} n &= a_{11} + \cdots + a_{1k} \\ &= a_{21} + \cdots + a_{2k} = \cdots = a_{k-1,1} + \cdots + a_{k-1,k}, \end{aligned}$$

where

$$(2)\qquad a_{11}\cdot a_{12}\cdot \cdots \cdot a_{1k} = a_{21}\cdot a_{22}\cdot \cdots \cdot a_{2k} = \cdots = a_{k-1,1}\cdot \cdots \cdot a_{k-1,k}$$

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and, in addition, the integers a_{ij} are pairwise distinct. (*Example*: *If* $k=4$, $n=1000$, *we have* $1000=65+214+276+445=89+130+321+460=92+178+195+535$ *and* $65\cdot 214\cdot 276\cdot 445=89\cdot 130\cdot 321\cdot 460=92\cdot 178\cdot 195\cdot 535=2^3\cdot 3\cdot 5^2\cdot 13\cdot 23\cdot 89\cdot 107$.)

Let us remove the requirement that the integers a_{ij} be pairwise distinct and require only that the $k-1$ partitions be different. Let $N^*(k)$ be the smallest integer satisfying the remaining conditions imposed upon $N(k)$ in the theorem. It is not difficult to show, employing the same line of thought as in the proof of the theorem, that $N^*(3)=19$ and $N(3)=23$. We omit the details. The values of $N^*(k)$ and $N(k)$ for $k>3$ are not known.

Some integers have three different partitions into 3 parts with equal products (e.g., $90=6+40+44=11+15+64=8+22+60$; $6\cdot 40\cdot 44=11\cdot 15\cdot 64=8\cdot 22\cdot 60=2^6\cdot 3\cdot 5\cdot 11$). We do not know if every sufficiently large integer has this property. More generally, we do not know if in our theorem, the number of different partitions of n may be taken larger than $k-1$.

PROOF OF THE THEOREM. Let n be a positive integer and consider the following system of $k-1$ linear equations in k unknowns:

$$
\begin{aligned}
&\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k = n\\
&\alpha_2 x_1 + \alpha_3 x_2 + \cdots + \alpha_k x_{k-1} + \alpha_1 x_k = n\\
&\quad\vdots\\
&\alpha_i x_1 + \alpha_{i+1} x_2 + \cdots + \alpha_{i-1} x_k = n\\
&\quad\vdots\\
&\alpha_{k-1} x_1 + \alpha_k x_2 + \cdots + \alpha_{k-2} x_k = n
\end{aligned}
\tag{3}
$$

or, more briefly:

$$\sum_{j=1}^{k} \alpha_{(i+j-2)} x_j = n, \qquad i = 1, 2, \cdots, k-1,$$

where (r) denotes the least positive residue of r modulo k. The coefficients α_j are pairwise distinct positive integers which will be chosen later. It is important to remark that their choice will be independent of n.

If we let $x_1, x_2, \cdots, x_k$ be a solution of (3) and set

$$a_{ij} = \alpha_{(i+j-1)} x_j, \tag{4}$$

we have a solution of (1) and (2). Of course, we desire a solution in which the numbers a_{ij}, given by (4), are pairwise distinct positive integers. We shall demonstrate that such a solution will exist if n is sufficiently large.

Let D_i be $(-1)^i$ times the determinant of the square matrix which is obtained by removing the ith column of the matrix of coefficients of (3). Then the system (3) has the solution

$$x_i = \frac{n + D_i s}{\sigma}, \qquad i = 1, 2, \cdots, k, \tag{5}$$

where s is a parameter and $\sigma = \sum_{j=1}^{k} \alpha_j$.

The remainder of the argument rests upon two lemmas, which we shall prove in subsequent sections.

LEMMA 1.

$$D_i \equiv D_j \pmod{\sigma}, \qquad i = 1, \cdots, k, \; j = 1, \cdots, k. \tag{6}$$

LEMMA 2. *It is possible to select* $\alpha_1, \alpha_2, \cdots, \alpha_k$ *so that*

$$(D_1, \sigma) = 1 \tag{7}$$

and

$$D_i = D_j \quad \textit{if and only if } i = j. \tag{8}$$

It follows from (5) and (7) that if $\alpha_1, \alpha_2, \cdots, \alpha_k$ are chosen as in Lemma 2 and if s is an integer satisfying

$$n + D_1 s \equiv 0 \pmod{\sigma}, \tag{9}$$

then $x_1, x_2, \cdots, x_k$ will be integers. It remains to show that we may determine s so that $x_1, x_2, \cdots, x_k$ will be positive integers and the integers a_{ij} will be pairwise distinct.

Now $a_{gh} = a_{ij}$ implies, by (4), that $\alpha_p x_h = \alpha_q x_j$ for suitable positive integers p and q not exceeding k. This implies by (5), that

$$\alpha_p(n + D_h s) = \alpha_q(n + D_j s). \tag{10}$$

Since the α's and the D's are pairwise distinct (Lemma 2), equation (10) has at most one solution unless $p = q$ and $h = j$, i.e., unless $g = i$ and $h = j$. There are at most $k^3(k-1)/2$ different equations of the form (10) with $p \neq q$, since p, q, h and j run from 1 to k. Hence there are at most $k^4/2$ values of s for which the integers a_{ij} will not be pairwise distinct. Now s is constrained by (9) to belong to a certain residue class modulo σ. Hence there is a positive value of s, say s_0, which satisfies (9), is less than $\sigma(k^4/2+1)$, and yields pairwise distinct integers a_{ij}.

Let $D=\max|D_i|$. If $n>\sigma(k^4/2+1)D$, it follows from (5) that, with $s=s_0$, $x_1, x_2, \cdots, x_k$ will be positive and hence the pairwise distinct integers a_{ij} will be positive. Since σ and D are independent of n, we may take $N(k)\leqq\sigma(k^4/2+1)D$. Our theorem is proved.

PROOF OF LEMMA 1. Border the matrix of coefficients of (3) with a row at the top having -1 in the ith column, $+1$ in the jth column and 0 elsewhere. Clearly the determinant of this $k\times k$ square matrix is D_i-D_j. On the other hand, by adding to the ith column of this matrix the sum of the remaining columns, we obtain a matrix with the same determinant whose ith column consists of one zero and $k-1$ σ's. Expanding by cofactors of the ith column, we see that $D_i-D_j\equiv 0 \pmod{\sigma}$.

PROOF OF LEMMA 2. We show first that if $\alpha_1, \alpha_2, \cdots, \alpha_{k-2}$ are any distinct positive integers, we can select each of the positive integers α_{k-1} and α_k in infinitely many ways so that they will be distinct from each other and from $\alpha_1, \alpha_2, \cdots, \alpha_{k-2}$ and so that (7) is satisfied.

If $\alpha_1, \alpha_2, \cdots, \alpha_{k-1}$ are prescribed, then $D_1=P_1(\alpha_k)$ is a polynomial of degree $k-1$ in α_k with integral coefficients. Let $\sigma'=\sigma-\alpha_k$. We may regard $P_1(\alpha_k)=P_1(\sigma-\sigma')=Q(\sigma)$ as a polynomial in σ with integral coefficients. Then $(D_1, \sigma)=(Q(\sigma), \sigma)=(Q(0), \sigma)=(P_1(-\sigma'), \sigma)$. If $P_1(-\sigma')\neq 0$, we can find infinitely many positive integral values of σ, and hence of α_k so that $(P_1(-\sigma'), \sigma)=1$.

If $\alpha_1, \alpha_2, \cdots, \alpha_{k-2}$ are prescribed, then $P_1(-\sigma')$ is a polynomial in σ' of degree $k-1$. This follows from the readily verified observation that if D_1 is considered as a polynomial in α_{k-1} and α_k, the only term of total degree $k-1$ is $\pm\alpha_k^{k-1}$. Hence, apart from at most $k-1$ integral values of σ', and hence of α_{k-1}, $P_1(-\sigma')\neq 0$, and the conclusion follows. Clearly, the selection of α_{k-1} and α_k can be made to satisfy the additional distinctness requirement.

If now $\alpha_1, \alpha_2, \cdots, \alpha_{k-1}$ are fixed, we may put $D_j=P_j(\alpha_k)$, $j=1, \cdots, k$ where, for $j>1$, the polynomials $P_j(\alpha_k)$ have leading term $\pm\alpha_{(k-j+1)}\alpha_k^{k-2}$. No two of the polynomials $P_j(\alpha_k)$ are identically equal since they have different leading terms. Hence the finitely many equations $P_i(\alpha_k)=P_j(\alpha_k)$, $i\neq j$, have each just a finite number of roots. From the infinite set of positive integral values of α_k which satisfy (7) and the distinctness requirement, we may choose one which is not a root of any of these equations. For this choice of α_k, (8) will also be satisfied.

ARIZONA STATE UNIVERSITY