algebraic theorem can be proved in its full generality only by resorting to the geometric lemma. This is in direct contrast to the situation where one can prove the geometric theorem that a finite Desarguesian plane satisfies Pappus's theorem only by using the algebraic fact that a finite division ring is commutative.

References


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PARTITIONS WITH EQUAL PRODUCTS

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T. S. Motzkin has conjectured (oral communication) that every sufficiently large positive integer can be partitioned into three positive integral parts in two different ways so that the products of the integers in the two partitions are equal (e.g., $13 = 1 + 6 + 6 = 2 + 2 + 9$; $1 \cdot 6 \cdot 6 = 2 \cdot 2 \cdot 9 = 36$). In this note we prove a generalization of this conjecture.

Theorem. Let $k$ be an integer $\geq 3$. There exists an integer $N(k)$ such that every integer $n \geq N(k)$ can be partitioned into $k$ parts in $k - 1$ different ways:

$$n = a_{11} + \cdots + a_{1k}$$
$$= a_{21} + \cdots + a_{2k} = \cdots = a_{k-1,1} + \cdots + a_{k-1,k},$$

where

$$a_{11} \cdot a_{12} \cdots a_{1k} = a_{21} \cdot a_{22} \cdots a_{2k} = \cdots = a_{k-1,1} \cdots a_{k-1,k}$$

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and, in addition, the integers $a_{ij}$ are pairwise distinct. (Example: If $k = 4$, $n = 1000$, we have $1000 = 65 + 214 + 276 + 445 = 89 + 130 + 321 + 460 = 92 + 178 + 195 + 535$ and $65 \cdot 214 \cdot 276 \cdot 445 = 89 \cdot 130 \cdot 321 \cdot 460 = 92 \cdot 178 \cdot 195 \cdot 535 = 2^3 \cdot 3 \cdot 5^2 \cdot 13 \cdot 23 \cdot 89 \cdot 107$.)

Let us remove the requirement that the integers $a_{ij}$ be pairwise distinct and require only that the $k-1$ partitions be different. Let $N^*(k)$ be the smallest integer satisfying the remaining conditions imposed upon $N(k)$ in the theorem. It is not difficult to show, employing the same line of thought as in the proof of the theorem, that $N^*(3) = 19$ and $N(3) = 23$. We omit the details. The values of $N^*(k)$ and $N(k)$ for $k > 3$ are not known.

Some integers have three different partitions into 3 parts with equal products (e.g., $90 = 6 + 40 + 44 = 11 + 15 + 64 = 8 + 22 + 60$; $6 \cdot 40 \cdot 44 = 11 \cdot 15 \cdot 64 = 8 \cdot 22 \cdot 60 = 2^4 \cdot 3 \cdot 5 \cdot 11$). We do not know if every sufficiently large integer has this property. More generally, we do not know if in our theorem, the number of different partitions of $n$ may be taken larger than $k - 1$.

**Proof of the theorem.** Let $n$ be a positive integer and consider the following system of $k - 1$ linear equations in $k$ unknowns:

\begin{align*}
\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_k x_k &= n \\
\alpha_2 x_1 + \alpha_3 x_2 + \cdots + \alpha_k x_{k-1} + \alpha_{k-1} x_k &= n \\
\vdots \\
\alpha_i x_1 + \alpha_{i+1} x_2 + \cdots + \alpha_{i-1} x_k &= n \\
\vdots \\
\alpha_{k-1} x_1 + \alpha_k x_2 + \cdots + \alpha_{k-2} x_k &= n
\end{align*}

or, more briefly:

$$
\sum_{j=1}^{k} \alpha_{(i+j-2)} x_j = n, \quad i = 1, 2, \ldots, k - 1,
$$

where $(r)$ denotes the least positive residue of $r$ modulo $k$. The coefficients $\alpha_{ij}$ are pairwise distinct positive integers which will be chosen later. It is important to remark that their choice will be independent of $n$.

If we let $x_1, x_2, \ldots, x_k$ be a solution of (3) and set

$$
a_{ij} = \alpha_{(i+j-1)} x_j,
$$

we have a solution of (1) and (2). Of course, we desire a solution in which the numbers $a_{ij}$, given by (4), are pairwise distinct positive
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Let $D_i$ be $(-1)^i$ times the determinant of the square matrix which is obtained by removing the $i$th column of the matrix of coefficients of (3). Then the system (3) has the solution

$$x_i = \frac{n + D_is}{\sigma}, \quad i = 1, 2, \ldots, k,$$

where $s$ is a parameter and $\sigma = \sum_{j=1}^{k} \alpha_j$.

The remainder of the argument rests upon two lemmas, which we shall prove in subsequent sections.

**Lemma 1.**

$$D_i \equiv D_j \pmod{\sigma}, \quad i = 1, \ldots, k, \quad j = 1, \ldots, k.$$  

**Lemma 2.** It is possible to select $\alpha_1, \alpha_2, \ldots, \alpha_k$ so that

$$\begin{align*}
(D_1, \sigma) &= 1 \\
(D_i, \sigma) &= 1 \quad \text{if and only if } i = j.
\end{align*}$$

It follows from (5) and (7) that if $\alpha_1, \alpha_2, \ldots, \alpha_k$ are chosen as in Lemma 2 and if $s$ is an integer satisfying

$$n + D_is \equiv 0 \pmod{\sigma},$$

then $x_1, x_2, \ldots, x_k$ will be integers. It remains to show that we may determine $s$ so that $x_1, x_2, \ldots, x_k$ will be positive integers and the integers $a_{ij}$ will be pairwise distinct.

Now $a_{ih} = a_{ij}$ implies, by (4), that $\alpha_p x_h = \alpha_q x_j$ for suitable positive integers $p$ and $q$ not exceeding $k$. This implies by (5), that

$$\alpha_p (n + D_h s) = \alpha_q (n + D_j s).$$

Since the $\alpha$'s and the $D$'s are pairwise distinct (Lemma 2), equation (10) has at most one solution unless $p = q$ and $h = j$, i.e., unless $g = i$ and $h = j$. There are at most $k^k(k-1)/2$ different equations of the form (10) with $p \neq q$, since $p, q, h$ and $j$ run from 1 to $k$. Hence there are at most $k^k/2$ values of $s$ for which the integers $a_{ij}$ will not be pairwise distinct. Now $s$ is constrained by (9) to belong to a certain residue class modulo $\sigma$. Hence there is a positive value of $s$, say $s_0$, which satisfies (9), is less than $\sigma(k^k/2 + 1)$, and yields pairwise distinct integers $a_{ij}$. 
Let $D = \max |D_i|$. If $n > \sigma(k^4/2+1)D$, it follows from (5) that, with $s = s_0, x_1, x_2, \ldots, x_k$ will be positive and hence the pairwise distinct integers $a_i$ will be positive. Since $\sigma$ and $D$ are independent of $n$, we may take $N(k) \leq \sigma(k^4/2+1)D$. Our theorem is proved.

**Proof of Lemma 1.** Border the matrix of coefficients of (3) with a row at the top having $-1$ in the $i$th column, $+1$ in the $j$th column and $0$ elsewhere. Clearly the determinant of this $k \times k$ square matrix is $D_i - D_j$. On the other hand, by adding to the $i$th column of this matrix the sum of the remaining columns, we obtain a matrix with the same determinant whose $i$th column consists of one zero and $k-1 \sigma$'s. Expanding by cofactors of the $i$th column, we see that $D_i - D_j \equiv 0 \pmod{\sigma}$.

**Proof of Lemma 2.** We show first that if $a_1, a_2, \ldots, a_{k-2}$ are any distinct positive integers, we can select each of the positive integers $a_{k-1}$ and $a_k$ in infinitely many ways so that they will be distinct from each other and from $a_1, a_2, \ldots, a_{k-2}$ and so that (7) is satisfied.

If $a_1, a_2, \ldots, a_{k-1}$ are prescribed, then $D_i = P_i(a_k)$ is a polynomial of degree $k-1$ in $a_k$ with integral coefficients. Let $\sigma' = \sigma - a_k$. We may regard $P_i(a_k) = P_i(\sigma - \sigma') = Q(\sigma')$ as a polynomial in $\sigma'$ with integral coefficients. Then $(D_i, \sigma) = (Q(\sigma), \sigma) = (Q(0), \sigma) = (P_i(-\sigma'), \sigma)$. If $P_i(-\sigma') \neq 0$, we can find infinitely many positive integral values of $\sigma'$, and hence of $a_k$ so that $(P_i(-\sigma'), \sigma) = 1$.

If $a_1, a_2, \ldots, a_{k-2}$ are prescribed, then $P_i(-\sigma')$ is a polynomial in $\sigma'$ of degree $k-1$. This follows from the readily verified observation that if $D_1$ is considered as a polynomial in $a_{k-1}$ and $a_k$, the only term of total degree $k-1$ is $\pm a_k^{k-1}$. Hence, apart from at most $k-1$ integral values of $\sigma'$, and hence of $a_{k-1}$, $P_i(-\sigma') \neq 0$, and the conclusion follows. Clearly, the selection of $a_{k-1}$ and $a_k$ can be made to satisfy the additional distinctness requirement.

If now $a_1, a_2, \ldots, a_{k-1}$ are fixed, we may put $D_i = P_j(a_k)$, $j = 1, \ldots, k$ where, for $j > 1$, the polynomials $P_j(a_k)$ have leading term $\pm a_{(k-j+1)}a_k^{k-2}$. No two of the polynomials $P_j(a_k)$ are identically equal since they have different leading terms. Hence the finitely many equations $P_i(a_k) = P_j(a_k)$, $i \neq j$, have each just a finite number of roots. From the infinite set of positive integral values of $a_k$ which satisfy (7) and the distinctness requirement, we may choose one which is not a root of any of these equations. For this choice of $a_k$, (8) will also be satisfied.

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