THE HADAMARD THEOREM FOR PERMANENTS

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1. **Statement of the result.** Let $A = (a_{ij})$ be an n-square non-negative hermitian matrix. A classical inequality of Hadamard states that

$$\det A \leqq \prod_{i=1}^{n} a_{ii} \tag{1}$$

with equality if and only if A has a zero row (and column) or A is a diagonal matrix: $A = \operatorname{diag}(a_{11}, \cdots, a_{nn})$. Several generalizations are known (e.g. [1]).

Let per A denote the permanent of A,

$$\operatorname{per} A = \sum_{\sigma} \prod_{i=1}^{n} a_{i\sigma(i)}, \tag{2}$$

where the summation extends over the whole symmetric group of degree n. It was conjectured in [4] that in analogy with (1),

$$\operatorname{per} A \geqq \prod_{i=1}^{n} a_{ii} \tag{3}$$

with the conditions of equality precisely those in the Hadamard determinant theorem. L. Mirsky recently listed this conjecture among several other problems concerning the permanent function [5]. This conjecture was suggested by an inequality of I. Schur [6] (see also [4]):

$$\operatorname{per} A \geqq \det A.$$

In an unsuccessful attempt [3] to prove (3) H. Minc and the present author obtained an inequality of the form

$$\operatorname{per} A \geqq c_n \prod_{i=1}^{n} a_{ii}$$

in which the constant c_n depends only on n and not on A.

It is the purpose of this paper to present the proof of an inequality that substantially generalizes (3) and to discuss the somewhat delicate cases of equality. Let $A(i)$ denote the principal submatrix of A obtained by deleting row and column i of A. The main result is contained in the following

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THEOREM 1. *Let $A=(a_{ij})$ be an $(r+1)$-square non-negative hermitian matrix. Then*

$$(4)\qquad (r+1)a_{ii}\operatorname{per} A(i) \geqq \operatorname{per} A \geqq a_{ii}\operatorname{per} A(i), \qquad 1\leqq i\leqq n.$$

If A has a zero row then (4) *is equality throughout. If A has no zero row then the lower equality holds if and only if a_{ii} is the only nonzero entry in row and column i of A; the upper equality holds if and only if the rank of A is* 1.

The permanent is unaltered by pre- and post-multiplication by permutation matrices so that we can take $i=1$ in proving (4).

Once (4) is established it is clear that an obvious induction on r will yield

THEOREM 2. *If $A=(a_{ij})$ is an n-square non-negative hermitian matrix then*

$$(5)\qquad \operatorname{per} A \geqq \prod_{i=1}^{n} a_{ii}$$

with equality if and only if A has a zero row or A is a diagonal matrix.

2. **Preliminaries.** Let U be an n-dimensional unitary space with inner product (x, y). For $1\leqq r\leqq n$ define $U^{(r)}$ to be the space of r-tensors on U [2, Chapter 7]; that is, $U^{(r)}$ is the dual space of the space $M_r(U)$ of all multilinear functionals of r-tuples of vectors from U. If $x_1, \cdots, x_r$ are in U then their *tensor product* $x_1\otimes \cdots \otimes x_r\in U^{(r)}$ is defined by

$$x_1\otimes\cdots\otimes x_r(\phi) = \phi(x_1, \cdots, x_r), \qquad \phi\in M_r(U).$$

An inner product in $U^{(r)}$ is given by

$$(6)\qquad (x_1\otimes\cdots\otimes x_r, y_1\otimes\cdots\otimes y_r) = \prod_{i=1}^{r}(x_i, y_i).$$

Define the completely symmetric operator $S^{(r)}: U^{(r)}\to U^{(r)}$ by

$$(7)\qquad S^{(r)}(x_1\otimes\cdots\otimes x_r) = \frac{1}{r!}\sum_{\sigma\in S_r} x_{\sigma(1)}\otimes\cdots\otimes x_{\sigma(r)},$$

where S_r is the symmetric group of degree r. The *symmetric product* of $x_1, \cdots, x_r$ is then defined by

$$(8)\qquad x_1\cdots x_r = S^{(r)}(x_1\otimes\cdots\otimes x_r).$$

The range space of $S^{(r)}$ is the symmetry class of completely symmetric tensors on U denoted by $U_{(r)}$ and since it is a subspace of $U^{(r)}$ it is unitary. By combining (6), (7), (8) we compute that the inner product of two symmetric products is

$$(9)\qquad (x_1 \cdots x_r, y_1 \cdots y_r)_r = \frac{1}{r!} \operatorname{per}((x_i, y_j)), \qquad 1 \leqq i, j \leqq r.$$

(The formula (9) is an immediate consequence of the fact that $S^{(r)}$ is idempotent and hermitian.) Next let $G_{r,n}$ denote the totality of non-decreasing sequences of length r chosen from $1, \cdots, n$. If $\omega \in G_{r,n}$ let $\mu(\omega)$ be the product of the factorials of the multiplicities of the distinct integers in ω; e.g. $\mu(3, 3, 7, 7, 7, 9) = 2!3!$. If $e_1, \cdots, e_n$ is an orthonormal (o.n.) basis of U then the

$$\binom{n+r-1}{r}$$

symmetric products $\sqrt{(r!/\mu(\omega))}e_{\omega_1} \cdots e_{\omega_r}$, $(\omega_1, \cdots, \omega_r) = \omega \in G_{r,n}$, constitute an o.n. basis of $U_{(r)}$ in the inner product $(\ ,\ )_r$. We let $e_\omega = e_{\omega_1} \cdots e_{\omega_r}$.

3. **Proofs.** Assume that A is $(r+1)$-square and has no zero row and let $D = \operatorname{diag}(a_{11}^{-1/2}, 1, \cdots, 1)$. Then the 1, 1 entry of $B = DAD$ is 1, per $B = a_{11}^{-1}$ per A, $A(1) = B(1)$, and B is also non-negative hermitian. If we prove Theorem 1 for B we will clearly have the result we want for A. Since B is non-negative hermitian it follows that B is a Gram matrix based on some set $e_1, v_1, \cdots, v_r$ where e_1 is a unit vector:

$$\begin{aligned} b_{11} &= 1 = (e_1, e_1), \\ b_{1,j+1} &= \bar{b}_{j+1,1} = (e_1, v_j), \qquad j = 1, \cdots, r, \\ b_{i+1,j+1} &= (v_i, v_j), \qquad i, j = 1, \cdots, r. \end{aligned}$$

Let $e_2, \cdots, e_n$ be a completion of e_1 to an o.n. basis of U. Define a map $T: U_{(r)} \to U_{(r+1)}$ by $T(e_\omega) = e_1 \cdot e_{\omega_1} \cdots e_{\omega_r}$, all $\omega = (\omega_1, \cdots, \omega_r) \in G_{r,n}$, and extend linearly. It is an easy consequence of the symmetry and linearity of the symmetric product in its factors that

$$T(x_1 \cdots x_r) = e_1 \cdot x_1 \cdots x_r$$

for all $x_1, \cdots, x_r$ in U. Let $R \subset U_{(r+1)}$ denote the range of T and let T^* denote the conjugate dual map of T; $T^*: R \to U_{(r)}$. That is, T^* satisfies

$$(Th, g)_{r+1} = (h, T^*g)_r \qquad \text{for all } h \in U_{(r)}, g \in R.$$

Next let $f(x_1, \cdots, x_r)$ denote the Rayleigh quotient

$$
\begin{aligned}
f(x_1, \cdots, x_r) &= \frac{(Tx_1 \cdots x_r, Tx_1 \cdots x_r)_{r+1}}{(x_1 \cdots x_r, x_1 \cdots x_r)_r} \\
&= \frac{(T^*Tx_1 \cdots x_r, x_1 \cdots x_r)_r}{(x_1 \cdots x_r, x_1 \cdots x_r)_r}.
\end{aligned} \tag{10}
$$

Now let H denote the non-negative hermitian transformation T^*T: $U_{(r)} \rightarrow U_{(r)}$. It is easy to verify [4, Theorem 3] that $x_1 \cdots x_r = 0$ if and only if some $x_i = 0$. Thus $f(x_1, \cdots, x_r)$ is defined for all sets of r nonzero vectors $x_1, \cdots, x_r$ in U and it is known that such values of f lie in the interval between the largest and smallest eigenvalues of H. We compute the eigenvalues of H by obtaining a matrix representation of H. The basis $\sqrt{(r!/\mu(\omega))}e_\omega$, $\omega \in G_{r,n}$, ordered lexicographically in ω, is o.n. for $U_{(r)}$ and the (τ, ω) entry of the matrix representation of H in this ordered basis is

$$
\frac{r!}{\sqrt{(\mu(\omega)\mu(\tau))}} (He_\omega, e_\tau)_r. \tag{11}
$$

Now

$$
\begin{aligned}
(He_\omega, e_\tau)_r &= (T^*Te_\omega, e_\tau)_r = (Te_\omega, Te_\tau)_{r+1} \\
&= (e_1 \cdot e_{\omega_1} \cdots e_{\omega_r}, e_1 \cdot e_{\tau_1} \cdots e_{\tau_r}) \\
&= (e_{(1,\omega)}, e_{(1,\tau)})_{r+1}
\end{aligned} \tag{12}
$$

where $(1, \omega)$ is the sequence $(1, \omega_1, \cdots, \omega_r) \in G_{r+1,n}$ and similarly for $(1, \tau)$. The vectors $\sqrt{((r+1)!/\mu(\alpha))}e_\alpha$, $\alpha \in G_{r+1,n}$, are an o.n. basis for $U_{(r+1)}$ in the inner product $(\ , \ )_{r+1}$. Moreover $(1, \omega) = (1, \tau)$ if and only if $\omega = \tau$. Hence from (12) we have

$$
(He_\omega, e_\tau)_r = \frac{\mu((1, \omega))}{(r+1)!} \delta_{\omega,\tau}.
$$

Thus the matrix representation of H in the basis $\sqrt{(r!/\mu(\omega))}e_\omega$ is diagonal and the eigenvalues of H are seen from (11) to be

$$
\begin{aligned}
\lambda_\omega(H) &= \frac{r!}{\mu(\omega)} (He_\omega, e_\omega)_r \\
&= \frac{r!}{\mu(\omega)} \frac{\mu((1, \omega))}{(r+1)!} \\
&= \frac{1}{r+1} \frac{\mu((1, \omega))}{\mu(\omega)}.
\end{aligned}
$$

Clearly $\mu((1,\omega))\geqq\mu(\omega)$ with equality if and only if $\omega_1>1$. Thus the minimum eigenvalue of H is $1/(r+1)$ and the symmetric tensors e_ω, $\omega_1>1$, constitute the totality of corresponding eigenvectors. Suppose the multiplicities greater than 1 of the distinct integers in ω are $m_1, \cdots, m_p$. Then the multiplicities for $(1,\omega)$ are either $m_1, \cdots, m_p$ or $m_1+1, m_2, \cdots, m_p$. In the first instance $\omega_1>1$ and $\mu((1,\omega))/\mu(\omega)=1$; in the second $\omega_1=1$ and $\mu((1,\omega))/\mu(\omega)=m_1+1$. This latter expression is maximal only when $m_1=r$, i.e., for the sequence $\omega=(1,\cdots,1)$. Thus we conclude that

$$1 \geqq f(x_1, \cdots, x_r) \geqq \frac{1}{r+1}\cdot \tag{13}$$

The lower equality holds if and only if $x_1\cdots x_r$ lies in the space spanned by the tensors e_ω, $\omega_1>1$, $\omega\in G_{r,n}$. The upper equality holds if and only if $x_1\cdots x_r$ is a multiple of $e_1\cdots e_1$. Now by (10) we have

$$\begin{aligned} f(v_1,\cdots,v_r) &= \frac{(Tv_1\cdots v_r, Tv_1\cdots v_r)_{r+1}}{(v_1\cdots v_r, v_1\cdots v_r)_r} \\ &= \frac{(e_1\cdot v_1\cdots v_r, e_1\cdot v_1\cdots v_r)_{r+1}}{(v_1\cdots v_r, v_1\cdots v_r)_r} \\ &= \frac{\dfrac{1}{(r+1)!}\operatorname{per} B}{\dfrac{1}{r!}\operatorname{per} B(1)} \\ &= \frac{1}{r+1}\,\frac{\operatorname{per} B}{\operatorname{per} B(1)} \end{aligned}$$

and it follows from (13) that

$$(r+1)\operatorname{per} B(1) \geqq \operatorname{per} B \geqq \operatorname{per} B(1). \tag{14}$$

As we indicated earlier (4) follows from (14).

We noted following (13) that the lower equality can hold if and only if

$$v_1\cdots v_r = \sum c_\omega e_\omega \tag{15}$$

where the summation extends over all $\omega\in G_{r,n}$ for which $\omega_1>1$. We prove that (15) implies that $(v_i, e_1)=0$, $i=1,\cdots,r$. Let h denote the tensor on the right side of (15). Then

$$(v_1 \cdots v_r, e_1 \cdots e_1)_r = \prod_{k=1}^{r} (v_k, e_1)$$

and

$$(h, e_1 \cdots e_1)_r = \sum_{\omega_1 > 1} c_\omega(e_\omega, e_1 \cdots e_1)_r = 0.$$

Hence some $(v_j, e_1) = 0$ and from the symmetry of the symmetric product we can assume $j = 1$. Suppose we have proved that $(v_1, e_1) = \cdots = (v_k, e_1) = 0$. Since no $v_i = 0$, $i = 1, \cdots, r$, we know [4, Theorem 3] that $0 \neq v_1 \cdots v_k \in U_{(k)}$. The tensors e_α, $\alpha \in G_{k,n}$, constitute a basis for $U_{(k)}$ and because $v_1 \cdots v_k \neq 0$ there exists an $\alpha \in G_{k,n}$ for which $(v_1 \cdots v_k, e_\alpha)_k \neq 0$. Let $\beta = (1, \cdots, 1, \alpha_1, \cdots, \alpha_k) \in G_{r,n}$ where the notation means that α has been preceded with $r-k$ 1's to make a nondecreasing sequence of length r. Now once again $(h, e_\beta) = 0$ and hence

$$0 = (v_1 \cdots v_r, e_\beta)_r = (v_1 \cdots v_k \cdot v_{k+1} \cdots v_r, e_1 \cdots e_1 \cdot e_{\alpha_1} \cdots e_{\alpha_k})_r$$

$$= \frac{1}{r!} \operatorname{per} \left(\begin{array}{ccc|ccc} (v_1, e_1) & \cdots & (v_1, e_1) & (v_1, e_{\alpha_1}) & \cdots & (v_1, e_{\alpha_k}) \\ \vdots & & & \vdots & & \\ (v_k, e_1) & \cdots & (v_k, e_1) & (v_k, e_{\alpha_1}) & \cdots & (v_k, e_{\alpha_k}) \\ \hline (v_{k+1}, e_1) & \cdots & (v_{k+1}, e_1) & (v_{k+1}, e_{\alpha_1}) & \cdots & (v_{k+1}, e_{\alpha_k}) \\ \vdots & & & \vdots & & \\ (v_r, e_1) & \cdots & (v_r, e_1) & (v_r, e_{\alpha_1}) & \cdots & (v_r, e_{\alpha_k}) \end{array} \right).$$

The upper left block in this last matrix is $k \times (r-k)$, the upper right block is $k \times k$, the lower left is $(r-k) \times (r-k)$ and the lower right is $(r-k) \times k$. The upper left block consists of zeros and hence by the Laplace expansion theorem for permanents we have

$$0 = \operatorname{per} \begin{pmatrix} (v_1, e_{\alpha_1}) & \cdots & (v_1, e_{\alpha_k}) \\ \vdots & & \vdots \\ (v_k, e_{\alpha_1}) & \cdots & (v_k, e_{\alpha_k}) \end{pmatrix} \operatorname{per} \begin{pmatrix} (v_{k+1}, e_1) & \cdots & (v_{k+1}, e_1) \\ \vdots & & \vdots \\ (v_r, e_1) & \cdots & (v_r, e_1) \end{pmatrix}$$

$$= k!(v_1 \cdots v_k, e_\alpha)_k (r-k)! \prod_{j=k+1}^{r} (v_j, e_1).$$

Now $(v_1 \cdots v_k, e_\alpha)_k \neq 0$ and hence some $(v_j, e_1) = 0$, $j = k+1, \cdots, r$. We can assume $(v_{k+1}, e_1) = 0$. Thus we have proved $(v_i, e_1) = 0$, $i = 1, \cdots, r$. In terms of the matrix B this implies that $b_{1j} = \bar{b}_{j1} = 0$, $j = 2, \cdots, r+1$, and hence $B = 1 \dotplus B(1)$.

If the upper equality holds then $v_1 \cdots v_r$ was seen to be a multiple of $e_1 \cdots e_1$. Since we are assuming that A has no zero row it follows from [4, Theorem 3] that $v_i = d_i e_1$, $i = 1, \cdots, r$, and hence B is the Gram matrix based on the set $e_1, d_1e_1, \cdots, d_re_1$. This means that B and hence A has rank 1, say $A = (z_i\bar{z}_j)$, $i, j = 1, \cdots, r+1$. Of course, when A has this form then

$$\operatorname{per} A = (r+1)!\prod_{j=1}^{r+1} |z_j|^2, \qquad \operatorname{per} A(1) = r!\prod_{j=2}^{r+1} |z_j|^2$$

and clearly per $A = (r+1)a_{11}$ per $A(1)$. This completes the proof of Theorem 1.

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