1. Introduction. Let \( J_\nu(z) \) be the Bessel function of the first kind, and denote the zeros of \( e^{-z}J_\nu(z) \) by \( j_{\nu,m} \), \( m = 1, 2, \ldots \), where \( |\text{Re}(j_{\nu,m})| \leq |\text{Re}(j_{\nu,m+1})| \). The Rayleigh function \( \sigma_{2n}(\nu) \) and the polynomial \( \phi_{2n}(\nu) \) of order \( 2n \) are defined by

\[
\sigma_{2n}(\nu) = \sum_{m=1}^{\infty} (j_{\nu,m})^{-2n}, \quad n = 1, 2, 3, \ldots;
\]

\[
\phi_{2n}(\nu) = 4^n \prod_{k=1}^{n} (\nu + k) \frac{[n/k]}{[n/k]} \sigma_{2n}(\nu),
\]

where \([x]\) denotes the greatest integer \( \leq x \). Rayleigh functions and polynomials of odd orders are identically zero. The function \( \sigma_{2n}(\nu) \) has been the subject of a number of investigations \([3, p. 502]\). Properties of \( \sigma_{2n}(\nu) \) have been discussed in a previous paper \([1]\).

The object of the present paper is to discuss properties of the polynomial \( \phi_{2n}(\nu) \). In this discussion we shall need three results from \([1]\). Formulas (3), (4) and (5) of the present paper are taken from \([1]\), and repeated here for the convenience of the reader.

\[
(3) \quad zJ_{\nu+1}(z)J_{\nu-1}(z) = 2 \sum_{n=1}^{\infty} \sigma_{2n}(\nu) z^{2n},
\]

\[
(4) \quad (\nu + n)\sigma_{2n}(\nu) = \sum_{k=1}^{n-1} \sigma_{2k}(\nu) \sigma_{2n-2k}(\nu),
\]

\[
(5) \quad \sum_{k=1}^{n} (-1)^{k-1}4^k(k!)^2 \binom{n}{k} \binom{\nu + n}{k} \sigma_{2k}(\nu) = n.
\]

2. Polynomials. The Rayleigh polynomials may be constructed from (3), (4) or (5) by using (2). We list here the first eight polynomials:

\[
\phi_{0}(\nu) = 1, \quad \phi_{4}(\nu) = 1, \quad \phi_{8}(\nu) = 2;
\]

\[
\phi_{8}(\nu) = 5\nu + 11, \quad \phi_{10}(\nu) = 14\nu + 38;
\]

\[
\phi_{12}(\nu) = 42\nu^3 + 362\nu^2 + 1026\nu + 946;
\]

\[
\phi_{14}(\nu) = 132\nu^3 + 1316\nu^2 + 4324\nu + 4580;
\]

\[
\phi_{18}(\nu) = 429\nu^5 + 7640\nu^4 + 53752\nu^3 + 185430\nu^2 + 311387\nu + 202738.
\]

Received by the editors January 30, 1963 and, in revised form, August 1, 1963.
For the next four polynomials the reader is referred to [2].

3. **The degree.** Let \( d_n \) be the degree of \( \phi_{2n}(v) \). Then if we substitute \( (2) \) in \( (4) \) and use induction on \( n \) the following is obtained:

\[
(6) \quad d_n = 1 - 2n + \sum_{k=1}^{n} \left\lfloor \frac{n}{k} \right\rfloor
\]

(see [2]). Take \( d_{n-1} \) and \( d_n \), the degrees of two consecutive polynomials \( \phi_{2n-2}(v) \) and \( \phi_{2n}(v) \), then

\[
\begin{align*}
    d_n - d_{n-1} &= 1 - 2n + \sum_{k=1}^{\infty} \left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n - 1}{k} \right\rfloor - \left\lfloor \frac{1}{k} \right\rfloor \\
    &= -2 + \sum_{k=1}^{\infty} \left\{ \left\lfloor \frac{n}{k} \right\rfloor - \left\lfloor \frac{n - 1}{k} \right\rfloor \right\}.
\end{align*}
\]

However, the quantity on the right side equals the number of nontrivial divisors of \( n \); therefore,

\[
(7) \quad d_n - d_{n-1} = \text{the number of nontrivial divisors of } n.
\]

From (7) we get the following:

\[
(8) \quad d_p = d_{p-1}, \quad \text{where } p \text{ is a prime.}
\]

4. **Coefficients.** We introduce here a symbol \( \epsilon(r, k, n) \) which is defined by the following relation

\[
\epsilon(r, k, n) = \left\lfloor \frac{n}{r} \right\rfloor - \left\lfloor \frac{k}{r} \right\rfloor - \left\lfloor \frac{(n - k)}{r} \right\rfloor.
\]

It is seen that the value of \( \epsilon(r, k, n) \) is either 0 or 1. In particular,

\[
(9) \quad \epsilon(1, k, n) = \epsilon(k, k, n) = 0.
\]

Now suppose we write \( \phi_{2n}(v) \) as

\[
(10) \quad \phi_{2n}(v) = \sum_{k=0}^{d_n} a_{n,k}v^k,
\]

then the main property of the coefficients \( a_{n,k} \) of \( v^k \) may be stated in the following

**Theorem I.** *Every coefficient* \( a_{n,k} \) *of* \( v^k \) *in the polynomial* \( \phi_{2n}(v) \) *is a positive integer.*

**Proof.** Observe that (4) may be written as

\[
(11) \quad (v + n)\sigma_{2n}(v) = \sum_{k=1}^{\lfloor n/2 \rfloor} a_k \sigma_{2k}(v)\sigma_{2n-2k}(v),
\]
where

\[ \alpha_k = 2, \quad k < \lfloor n/2 \rfloor; \quad \alpha_{\lfloor n/2 \rfloor} = \begin{cases} 2 & \text{if } n \text{ is odd}, \\ 1 & \text{otherwise}. \end{cases} \]

Substitute (2) in (11). Then since

\[
\prod_{r=1}^{k} (v + r)^{-[k/r]} = \prod_{r=1}^{n-k} (v + r)^{-[k/r]}, \quad k \leq \lfloor n/2 \rfloor;
\]

\[
\prod_{r=n-k+1}^{n-1} (v + r)^{[n/r]} = \prod_{r=n-k+1}^{n-1} (v + r), \quad k \leq \lfloor n/2 \rfloor,
\]

we obtain the following:

\[
(12) \quad \phi_{2n}(v) = \sum_{k=1}^{\lfloor n/2 \rfloor} \alpha_k \prod_{r=2}^{\lfloor n/2 \rfloor} (v + r)^{s(r,k,n)} \prod_{r=n-k+1}^{n-1} (v + r)^{\phi_{2k}\phi_{2n-2k}(v)}.
\]

If we now consider (12) with induction on \( n \) the required result is obtained.

5. Real zeros. In this section we shall prove that all real zeros of \( \phi_{2n}(v) \) lie in a certain interval. First we see that in consideration of (12) and induction on \( n \) the following inequality is valid:

\[
(13) \quad \phi_{2n}(v) > 0, \quad \text{if } v \geq -2.
\]

We note that (13) gives an upper bound \(-2\) for the set of all real zeros of \( \phi_{2n}(v) \). Next we shall determine a lower bound for the same set, and then combine the results in a theorem. We begin with

\[
(14) \quad \phi_{2n}(-s) \neq 0, \quad \text{if } s \text{ is a positive integer and } s \mid n.
\]

To prove this, consider (3):

\[
2 \sum_{k=1}^{\infty} \sigma_{2k}(v)z^{2k} = zJ_{v+1}(z)J_{v-1}(z) \]

\[
= z \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k(z/2)^{r+1+2k}}{k! \Gamma(v+2+k)} \right\} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k(z/2)^{r+2k}}{k! \Gamma(v+1+k)} \right\}^{-1}
\]

\[
= \frac{1}{2} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+r}}{k!(r)_{k+1} 4^k} \right\} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{k! (r)_{k+1} 4^k} \right\}^{-1},
\]

where \((v)_m = (v + 1)(v + 2) \cdots (v + m)\), and \((v)_0 = 1\). Substitute \( z = t^{1/2s}(v+s)^{1/2s} \) in the above, then
\[ 4 \sum_{k=1}^{\infty} \sigma_{2k}(v) \frac{k^{1/s}(v+s)^{k/s}}{k!(v+k+1)^{4k}} = \left\{ \sum_{k=0}^{\infty} \frac{(-1)^{k} (k+1)^{1/s}(v+s)^{(k+1)/s}}{k!(v+k+1)^{4k}} \right\}^{-1} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^{k} k^{1/s}(v+s)^{k/s}}{k!(v+k)^{4k}} \right\}^{-1}. \]

Now if we let \( v \to -s \), then in view of (2) all terms of the left side, except those for which \( k \) is a multiple of \( s \), become zero. Similarly, all quantities on the right except one from the first series and two from the second vanish. Hence after a simplification we have

\[ \sum_{m=1}^{\infty} \lim_{v \to -s} (v+s)^m \sigma_{2m}(v) = \frac{s t}{s!(s-1)!4^s} \left\{ 1 - \frac{t}{s!(s-1)!4^s} \right\}^{-1}. \]

Equating the coefficients of \( t^n \) on two sides we get

\[ \lim_{v \to -s} (v+s)^m \sigma_{2m}(v) = s \{s!(s-1)!4^s\}^{-m}. \]

Clearly (14) is true in view of (2) and (15). In fact, an explicit expression for \( \sigma_{2m}(-s) \) can be obtained. In particular,

\[ \lim_{v \to -n} (v+n) \sigma_{2n}(v) = \{(n-1)!\}^{-2} (n-1)!; \]

\[ \phi_{2n}(-n) = \{(n-1)!\}^{-2} \prod_{k=1}^{n-1} (k-n)^{[n/k]}. \]

We shall now show that

\[ (v+n) \sigma_{2n}(v) > 0, \quad v \leq -n. \]

The proof of this inequality is by induction on \( n \). We observe that \( \sigma_2(v) < 0 \) if \( v < -1 \), and \( \sigma_4(v) < 0 \) if \( v < -2 \). Suppose \( \sigma_{2k}(v) < 0, v < -k \), for \( k < n \). Then \( \sigma_{2k}(v) \) and \( \sigma_{2n-2k}(v) \) are negative when \( v < -k \) and \( v < -(n-k) \), respectively; so, in particular, when \( v < -n \). Therefore, for \( k < n, \sigma_{2k}(v) \sigma_{2n-2k}(v) > 0 \) when \( v < -n \). Hence

\[ \sum_{k=1}^{n-1} \sigma_{2k}(v) \sigma_{2n-2k}(v) > 0, \quad v < -n. \]

Using (4) we obtain

\[ \sigma_{2n}(v) = (v+n)^{-1} \sum_{k=1}^{n-1} \sigma_{2k}(v) \sigma_{2n-2k}(v) < 0, \quad v < -n. \]

This result and (16) then lead to (18).

**Lemma.** If \( v \leq -n \), then \( \phi_{2n}(v) \) is positive or negative according as \( \lceil \sqrt{n} \rceil \) is odd or even.
Proof. Substitute (2) in (18), then
\[
\phi_{2n}(\nu) \prod_{k=1}^{n-1} (\nu + k)^{\lfloor n/k \rfloor} > 0, \quad \nu \leq -n.
\]
This implies that for \( \nu \leq -n \), \( \phi_{2n}(\nu) \) and the product \( \prod_{k=1}^{n-1} (\nu + k)^{\lfloor n/k \rfloor} \) have like signs. However, for \( \nu \leq -n \), the product is positive or negative according as the sum \( \sum_{k=1}^{n-1} \lfloor n/k \rfloor \) is even or odd; and this sum is even or odd according as \( \lfloor \sqrt{n} \rfloor \) is odd or even. This completes the proof.

The lemma implies that \( \phi_{2n}(\nu) \) does not have zeros \( \leq -n \). Consequently, we have the following

Theorem II. All real roots of \( \phi_{2n}(\nu) = 0 \) lie in the interval \((-n, -2)\).

6. Congruences. We intend to prove two congruences for the Rayleigh polynomials. The first is
\[
2 \sum_{k=1}^{s} (k - 1)! \left(\begin{array}{c}
\nu + n - 1 \\
n/2
\end{array}\right) \prod_{r=2}^{[n/2]} (\nu + r)^{s(r,k,n)} \phi_{2k}(\nu) \phi_{2n-2k}(\nu)
\equiv \phi_{2n}(\nu) \pmod{(\nu + n - 1)(\nu + n - 2) \cdots (\nu + n - s)},
\]
where \( s \leq [n/2] - 1 \). To prove this congruence take (12) and observe that
\[
\prod_{r=n-k+1}^{n-1} (\nu + r) = (k - 1)! \left(\begin{array}{c}
\nu + n - 1 \\
k - 1
\end{array}\right);
\]
then (12) may be written as
\[
\phi_{2n}(\nu) = \prod_{k=1}^{[n/2]} \alpha_k T_k,
\]
where
\[
T_k = (k - 1)! \left(\begin{array}{c}
\nu + n - 1 \\
n/2
\end{array}\right) \prod_{r=2}^{[n/2]} (\nu + r)^{s(r,k,n)} \phi_{2k}(\nu) \phi_{2n-2k}(\nu).
\]
Then since \( \alpha_k = 2 \) if \( k < [n/2] \), (20) may be written as
\[
\phi_{2n}(\nu) = 2 \sum_{k=1}^{s} T_k + \sum_{k=s+1}^{[n/2]} \alpha_k T_k, \quad s \leq \left\lfloor \frac{n}{2} \right\rfloor - 1.
\]
However,
\[
\sum_{k=s+1}^{[n/2]} \alpha_k T_k \equiv 0 \pmod{(\nu + n - 1)(\nu + n - 2) \cdots (\nu + n - s)}.
\]
Hence

\[ \phi_{2n}(v) \equiv 2 \sum_{k=1}^{s} T_k \pmod{(v + n - 1)(v + n - 2) \cdots (v + n - s)} \]

which is (19). Substitute \( s = 1 \) in (19), and observe that for \( r > 1 \), \( e(r, 1, n) \) equals 1 or 0 according as \( r \mid n \) or \( r \nmid n \). Then

\[ \phi_{2n}(v) \equiv 2 \prod_{\delta} (v + \delta) \phi_{2n-2}(v) \pmod{(v + n - 1)}, \]

where \( \delta \) ranges over the proper divisors of \( n \). From (21) we get the following

\[ \phi_{2n}(v) \equiv 2 \prod_{\delta} (\delta + 1 - n) \phi_{2n-2}(v) \pmod{(v + n - 1)}; \]

\[ \phi_{2n}(v) \equiv 2 \phi_{2n-2}(v) \pmod{(v + n - 1)}, \] if \( n \) is an odd prime.

The second congruence for the polynomials \( \phi_{2n}(v) \) is

\[ \sum_{k=1}^{s} \left\{ P_{2k}(v) + P_{2n-2k-2}(v) \right\} \]

\[ \equiv (-1)^{n(n - 1)} \phi_{2n}(v) \]

\[ + \prod_{r=1}^{\lfloor n/2 \rfloor} (v + r)^{(n/r) - 1} \pmod{(v + n - 1) \cdots (v + n - s)}, \]

where

\[ P_{2m}(v) = (-1)^{m-1}(m!)^2 \binom{n}{m} \binom{v + n}{m} \phi_{2m}(v) \prod_{r=1}^{\lfloor n/r \rfloor} (v + r)^{(n/r) - 1}, \]

and

\[ 0 \leq s < \lfloor n/2 \rfloor. \]

This congruence is obtained from (5). For, substitution of (2) in (5) yields

\[ \sum_{k=1}^{n} (-1)^{k-1}(k!)^2 \binom{n}{k} \binom{v + n}{k} \phi_{2k}(v) \prod_{s=1}^{k} (v + s)^{-[k/s]} = n. \]

Now multiply the above by

\[ \prod_{r=1}^{n} (v + r)^{(n/r) - 1}, \]

and observe that \( \lfloor n/r \rfloor - \lfloor k/r \rfloor - 1 \geq 0 \), for \( k \leq n - 1, r \leq n \). Then
\((-1)^{n-1}(n-1) \Phi_{n}(\nu) + \prod_{r=1}^{n} \nu + r \) \[\text{mod} \ (\nu + n)\]