THE RAYLEIGH POLYNOMIAL

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1. **Introduction.** Let $J_\nu(z)$ be the Bessel function of the first kind, and denote the zeros of $z^{-\nu}J_\nu(z)$ by $j_{\nu,m}$, $m=1, 2, \cdots$, where $|\mathrm{Re}(j_{\nu,m})| \leqq |\mathrm{Re}(j_{\nu,m+1})|$. The Rayleigh function $\sigma_{2n}(\nu)$ and the polynomial $\phi_{2n}(\nu)$ of order $2n$ are defined by

$$(1)\qquad \sigma_{2n}(\nu) = \sum_{m=1}^{\infty} (j_{\nu,m})^{-2n},$$

$$n = 1, 2, 3, \cdots;$$

$$(2)\qquad \phi_{2n}(\nu) = 4^n \prod_{k=1}^{n} (\nu + k)^{[n/k]}\sigma_{2n}(\nu),$$

where $[x]$ denotes the greatest integer $\leqq x$. Rayleigh functions and polynomials of odd orders are identically zero. The function $\sigma_{2n}(\nu)$ has been the subject of a number of investigations [**3**, p. 502]. Properties of $\sigma_{2n}(\nu)$ have been discussed in a previous paper [**1**].

The object of the present paper is to discuss properties of the polynomial $\phi_{2n}(\nu)$. In this discussion we shall need three results from [**1**]. Formulas (3), (4) and (5) of the present paper are taken from [**1**], and repeated here for the convenience of the reader.

$$(3)\qquad zJ_{\nu+1}(z)J_\nu^{-1}(z) = 2\sum_{n=1}^{\infty} \sigma_{2n}(\nu)z^{2n},$$

$$(4)\qquad (\nu + n)\sigma_{2n}(\nu) = \sum_{k=1}^{n-1} \sigma_{2k}(\nu)\sigma_{2n-2k}(\nu),$$

$$(5)\qquad \sum_{k=1}^{n} (-1)^{k-1}4^k(k!)^2 \binom{n}{k}\binom{\nu+n}{k}\sigma_{2k}(\nu) = n.$$

2. **Polynomials.** The Rayleigh polynomials may be constructed from (3), (4) or (5) by using (2). We list here the first eight polynomials:

$$\phi_2(\nu) = 1, \qquad \phi_4(\nu) = 1, \qquad \phi_6(\nu) = 2,$$

$$\phi_8(\nu) = 5\nu + 11, \qquad \phi_{10}(\nu) = 14\nu + 38,$$

$$\phi_{12}(\nu) = 42\nu^3 + 362\nu^2 + 1026\nu + 946,$$

$$\phi_{14}(\nu) = 132\nu^3 + 1316\nu^2 + 4324\nu + 4580,$$

$$\phi_{16}(\nu) = 429\nu^5 + 7640\nu^4 + 53752\nu^3 + 185430\nu^2 + 311387\nu + 202738.$$

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For the next four polynomials the reader is referred to [2].

3. **The degree.** Let d_n be the degree of $\phi_{2n}(\nu)$. Then if we substitute (2) in (4) and use induction on n the following is obtained:

$$d_n = 1 - 2n + \sum_{k=1}^{n} \left[\frac{n}{k}\right] \tag{6}$$

(see [2]). Take d_{n-1} and d_n, the degrees of two consecutive polynomials $\phi_{2n-2}(\nu)$ and $\phi_{2n}(\nu)$, then

$$\begin{aligned} d_n - d_{n-1} &= 1 - 2n + \sum_{k=1}^{\infty}\left[\frac{n}{k}\right] - \left\{1 - 2(n-1) + \sum_{k=1}^{\infty}\left[\frac{n-1}{k}\right]\right\} \\ &= -2 + \sum_{k=1}^{\infty}\left\{\left[\frac{n}{k}\right] - \left[\frac{n-1}{k}\right]\right\}. \end{aligned}$$

However, the quantity on the right side equals the number of nontrivial divisors of n; therefore,

$$d_n - d_{n-1} = \text{the number of nontrivial divisors of } n. \tag{7}$$

From (7) we get the following:

$$d_p = d_{p-1}, \quad \text{where } p \text{ is a prime.} \tag{8}$$

4. **Coefficients.** We introduce here a symbol $\epsilon(r, k, n)$ which is defined by the following relation

$$\epsilon(r, k, n) = [n/r] - [k/r] - [(n-k)/r].$$

It is seen that the value of $\epsilon(r, k, n)$ is either 0 or 1. In particular,

$$\epsilon(1, k, n) = \epsilon(k, k, n) = 0. \tag{9}$$

Now suppose we write $\phi_{2n}(\nu)$ as

$$\phi_{2n}(\nu) = \sum_{k=0}^{d_n} a_{n,k}\nu^k, \tag{10}$$

then the main property of the coefficients $a_{n,k}$ of ν^k may be stated in the following

THEOREM I. *Every coefficient $a_{n,k}$ of ν^k in the polynomial $\phi_{2n}(\nu)$ is a positive integer.*

PROOF. Observe that (4) may be written as

$$(\nu + n)\sigma_{2n}(\nu) = \sum_{k=1}^{[n/2]} \alpha_k \sigma_{2k}(\nu)\sigma_{2n-2k}(\nu), \tag{11}$$

where

$$\alpha_k = 2,\ k < [n/2]; \qquad \alpha_{[n/2]} = \begin{cases} 2 & \text{if } n \text{ is odd,} \\ 1 & \text{otherwise.} \end{cases}$$

Substitute (2) in (11). Then since

$$\prod_{r=1}^{k} (\nu + r)^{-[k/r]} = \prod_{r=1}^{n-k} (\nu + r)^{-[k/r]}, \qquad k \leqq [n/2];$$

$$\prod_{r=n-k+1}^{n-1} (\nu + r)^{[n/r]} = \prod_{r=n-k+1}^{n-1} (\nu + r), \qquad k \leqq [n/2],$$

we obtain the following:

$$\phi_{2n}(\nu) = \sum_{k=1}^{[n/2]} \alpha_k \prod_{r=2}^{[n/2]} (\nu + r)^{\epsilon(r,k,n)} \prod_{r=n-k+1}^{n-1} (\nu + r)\phi_{2k}\phi_{2n-2k}(\nu). \tag{12}$$

If we now consider (12) with induction on n the required result is obtained.

5. **Real zeros.** In this section we shall prove that all real zeros of $\phi_{2n}(\nu)$ lie in a certain interval. First we see that in consideration of (12) and induction on n the following inequality is valid:

$$\phi_{2n}(\nu) > 0, \quad \text{if} \quad \nu \geqq -2. \tag{13}$$

We note that (13) gives an upper bound -2 for the set of all real zeros of $\phi_{2n}(\nu)$. Next we shall determine a lower bound for the same set, and then combine the results in a theorem. We begin with

$$\phi_{2n}(-s) \neq 0, \text{ if } s \text{ is a positive integer and } s \mid n. \tag{14}$$

To prove this, consider (3):

$$\begin{aligned} 2\sum_{k=1}^{\infty} \sigma_{2k}(\nu) z^{2k} &= zJ_{\nu+1}(z)J_\nu^{-1}(z) \\ &= z\left\{\sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+1+2k}}{k!\Gamma(\nu+2+k)}\right\}\left\{\sum_{k=0}^{\infty} \frac{(-1)^k (z/2)^{\nu+2k}}{k!\Gamma(\nu+1+k)}\right\}^{-1} \\ &= \frac{1}{2}\left\{\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+2}}{k!(\nu)_{k+1}4^k}\right\}\left\{\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{k!(\nu)_k 4^k}\right\}^{-1}, \end{aligned}$$

where $(\nu)_m = (\nu+1)(\nu+2)\cdots(\nu+m)$, and $(\nu)_0 = 1$. Substitute $z = t^{1/2s}(\nu+s)^{1/2s}$ in the above, then

$$4 \sum_{k=1}^{\infty} \sigma_{2k}(\nu) t^{k/s} (\nu + s)^{k/s} = \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k t^{(k+1)/s} (\nu + s)^{(k+1)/s}}{k!(\nu)_{k+1} 4^k} \right\} \left\{ \sum_{k=0}^{\infty} \frac{(-1)^k t^{k/s} (\nu + s)^{k/s}}{k!(\nu)_k 4^k} \right\}^{-1}.$$

Now if we let $\nu \to -s$, then in view of (2) all terms of the left side, except those for which k is a multiple of s, become zero. Similarly, all quantities on the right except one from the first series and two from the second vanish. Hence after a simplification we have

$$\sum_{m=1}^{\infty} t^m \lim_{\nu \to -s} (\nu + s)^m \sigma_{2ms}(\nu) = \frac{st}{s!(s-1)!4^s} \left\{ 1 - \frac{t}{s!(s-1)!4^s} \right\}^{-1}.$$

Equating the coefficients of t^m on two sides we get

$$\lim_{\nu \to -s} (\nu + s)^m \sigma_{2ms}(\nu) = s\{s!(s-1)!4^s\}^{-m}. \tag{15}$$

Clearly (14) is true in view of (2) and (15). In fact, an explicit expression for $\phi_{2ms}(-s)$ can be obtained. In particular,

$$\lim_{\nu \to -n} (\nu + n)\sigma_{2n}(\nu) = \{(n-1)!\}^{-2} 4^{-n}; \tag{16}$$

$$\phi_{2n}(-n) = \{(n-1)!\}^{-2} \prod_{k=1}^{n-1} (k - n)^{[n/k]}. \tag{17}$$

We shall now show that

$$(\nu + n)\sigma_{2n}(\nu) > 0, \qquad \nu \leqq -n. \tag{18}$$

The proof of this inequality is by induction on n. We observe that $\sigma_2(\nu) < 0$ if $\nu < -1$, and $\sigma_4(\nu) < 0$ if $\nu < -2$. Suppose $\sigma_{2k}(\nu) < 0$, $\nu < -k$, for $k < n$. Then $\sigma_{2k}(\nu)$ and $\sigma_{2n-2k}(\nu)$ are negative when $\nu < -k$ and $\nu < -(n-k)$, respectively; so, in particular, when $\nu < -n$. Therefore, for $k < n$, $\sigma_{2k}(\nu)\sigma_{2n-2k}(\nu) > 0$ when $\nu < -n$. Hence

$$\sum_{k=1}^{n-1} \sigma_{2k}(\nu)\sigma_{2n-2k}(\nu) > 0, \qquad \nu < -n.$$

Using (4) we obtain

$$\sigma_{2n}(\nu) = (\nu + n)^{-1} \sum_{k=1}^{n-1} \sigma_{2k}(\nu)\sigma_{2n-2k}(\nu) < 0, \qquad \nu < -n.$$

This result and (16) then lead to (18).

LEMMA. *If $\nu \leqq -n$, then $\phi_{2n}(\nu)$ is positive or negative according as $[\sqrt{n}]$ is odd or even.*

PROOF. Substitute (2) in (18), then

$$\phi_{2n}(\nu) \prod_{k=1}^{n-1} (\nu + k)^{-[n/k]} > 0, \qquad \nu \leqq -n.$$

This implies that for $\nu \leqq -n$, $\phi_{2n}(\nu)$ and the product $\prod_{k=1}^{n-1} (\nu+k)^{-[n/k]}$ have like signs. However, for $\nu \leqq -n$, the product is positive or negative according as the sum $\sum_{k=1}^{n-1} [n/k]$ is even or odd; and this sum is even or odd according as $[\sqrt{n}]$ is odd or even. This completes the proof.

The lemma implies that $\phi_{2n}(\nu)$ does not have zeros $\leqq -n$. Consequently, we have the following

THEOREM II. *All real roots of* $\phi_{2n}(\nu)=0$ *lie in the interval* $(-n, -2)$.

6. **Congruences.** We intend to prove two congruences for the Rayleigh polynomials. The first is

$$\begin{aligned} &2 \sum_{k=1}^{s} (k-1)! \binom{\nu+n-1}{k-1} \prod_{r=2}^{[n/2]} (\nu+r)^{\epsilon(r,k,n)} \phi_{2k}(\nu)\phi_{2n-2k}(\nu) \\ &\qquad \equiv \phi_{2n}(\nu) \pmod{(\nu+n-1)(\nu+n-2)\cdots(\nu+n-s)}, \end{aligned} \tag{19}$$

where $s \leqq [n/2]-1$. To prove this congruence take (12) and observe that

$$\prod_{r=n-k+1}^{n-1} (\nu+r) = (k-1)! \binom{\nu+n-1}{k-1};$$

then (12) may be written as

$$\phi_{2n}(\nu) = \prod_{k=1}^{[n/2]} \alpha_k T_k, \tag{20}$$

where

$$T_k = (k-1)! \binom{\nu+n-1}{k-1} \prod_{r=2}^{[n/2]} (\nu+r)^{\epsilon(r,k,n)} \phi_{2k}(\nu)\phi_{2n-2k}(\nu).$$

Then since $\alpha_k=2$ if $k<[n/2]$, (20) may be written as

$$\phi_{2n}(\nu) = 2\sum_{k=1}^{s} T_k + \sum_{k=s+1}^{[n/2]} \alpha_k T_k, \qquad s \leqq \left[\frac{n}{2}\right] - 1.$$

However,

$$\sum_{k=s+1}^{[n/2]} \alpha_k T_k \equiv 0 \pmod{(\nu+n-1)(\nu+n-2)\cdots(\nu+n-s)}.$$

Hence

$$\phi_{2n}(\nu) \equiv 2 \sum_{k=1}^{s} T_k \pmod{(\nu + n - 1)(\nu + n - 2) \cdots (\nu + n - s)}$$

which is (19). Substitute $s=1$ in (19), and observe that for $r>1$, $\epsilon(r, 1, n)$ equals 1 or 0 according as $r|n$ or $r\nmid n$. Then

$$\phi_{2n}(\nu) \equiv 2 \prod_{\delta} (\nu + \delta)\phi_{2n-2}(\nu) \pmod{(\nu + n - 1)}, \tag{21}$$

where δ ranges over the proper divisors of n. From (21) we get the following

$$\phi_{2n}(\nu) \equiv 2 \prod_{\delta} (\delta + 1 - n)\phi_{2n-2}(\nu) \pmod{(\nu + n - 1)}; \tag{22}$$

$$\phi_{2n}(\nu) \equiv 2\phi_{2n-2}(\nu) \pmod{(\nu + n - 1)}, \text{ if } n \text{ is an odd prime.} \tag{23}$$

The second congruence for the polynomials $\phi_{2n}(\nu)$ is

$$\begin{aligned} n^{-1} \sum_{k=1}^{s} \{P_{2k}(\nu) + P_{2n-2s+2k-2}(\nu)\} \\ \equiv (-1)^n(n-1)!\phi_{2n}(\nu) \\ + \prod_{r=1}^{[n/2]} (\nu + r)^{[n/r]-1} \pmod{(\nu + n) \cdots (\nu + n - s)}, \end{aligned} \tag{24}$$

where

$$P_{2m}(\nu) = (-1)^{m-1}(m!)^2 \binom{n}{m}\binom{\nu+n}{m}\phi_{2m}(\nu) \prod_{r=1}^{[n/2]} (\nu + r)^{[n/r]-[m/r]-1},$$

and

$$0 \leqq s < [n/2].$$

This congruence is obtained from (5). For, substitution of (2) in (5) yields

$$\sum_{k=1}^{n} (-1)^{k-1}(k!)^2 \binom{n}{k}\binom{\nu+n}{k} \phi_{2k}(\nu) \prod_{s=1}^{k} (\nu + s)^{-[k/s]} = n.$$

Now multiply the above by

$$\prod_{r=1}^{n} (\nu + r)^{[n/r]-1},$$

and observe that $[n/r]-[k/r]-1 \geqq 0$, for $k \leqq n-1$, $r \leqq n$. Then

$$(-1)^n(n-1)!\phi_{2n}(\nu)+\prod_{r=1}^{n}(\nu+r)^{[n/r]-1}$$
$$=n^{-1}\sum_{k=1}^{n-1}(-1)^{k-1}(k!)^2\binom{n}{k}\binom{\nu+n}{k}\phi_{2k}(\nu)\prod_{r=1}^{n}(\nu+r)^{[n/r]-[k/r]-1}.$$

The above relation leads to the required congruence. If we set $s=0$ in (24) then

$$(25)\quad (-1)^{n-1}(n-1)!\phi_{2n}(\nu)\equiv\prod_{r=1}^{[n/2]}(\nu+r)^{[n/r]-1}\ (\mathrm{mod}\ (\nu+n))$$

or

$$(\nu+1)(\nu+2)\cdots(\nu+n-1)\phi_{2n}(\nu)$$
$$(26)\qquad \equiv\prod_{r=1}^{[n/2]}(\nu+r)^{[n/r]-1}\ (\mathrm{mod}\ (\nu+n)).$$

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