SHORTER NOTES

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THE MARCINKIEWICZ INTERPOLATION THEOREM

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We show that the Marcinkiewicz Theorem on the interpolation of operators acting on $L^p$ spaces (see [3, pp. 111-116]) is an immediate consequence of two easily proved inequalities. The first one is a well-known result of Hardy (see [1, pp. 245-246]):

If $q \geq 1$, $r > 0$, and $g$ is a measurable, non-negative function on $(0, \infty)$, then

$$
\left( \int_0^\infty \left( \int_0^t g(y) dy \right)^q t^{-r-1} dt \right)^{1/q} \leq \frac{q}{r} \left( \int_0^\infty (yg(y)) y^{-r-1} dy \right)^{1/q}
$$

and

$$
\left( \int_0^\infty \left( \int_t^\infty g(y) dy \right)^q t^{-r-1} dt \right)^{1/q} \leq \frac{q}{r} \left( \int_0^\infty (yg(y)) y^{-r-1} dy \right)^{1/q}.
$$

The second one can be found in [2]:

If $g$ is non-negative and nonincreasing on $(0, \infty)$, $1 \leq q_1 \leq q_2 \leq \infty$ and $1 \leq p \leq \infty$, then

$$
\left( \int_0^\infty \left[ t^{1/p} g(t) \right] q_2 \frac{dt}{t} \right)^{1/q_2} \leq \left( \frac{q_1}{p} \right)^{1/q_1-1/q_2} \left( \int_0^\infty \left[ t^{1/p} g(t) \right] q_1 \frac{dt}{t} \right)^{1/q_1}.
$$

If $h$ is measurable on a measure space $M$ with measure $m$, its distribution function is defined for $y>0$ by $\lambda_h(y) = \lambda(y) = m \{ x \in M ; f(x) > y \}$. The nonincreasing rearrangement of $h$ onto $(0, \infty)$ is then the function given by $h^*(t) = \inf \{ y > 0 ; \lambda(y) \leq t \}, t > 0$. Both $h^*$ and $\lambda$ are non-negative and nonincreasing functions that are continuous from the right. $h^*$ and $h$ have the same distribution function, thus $\|h^*\|_p = \|h\|_p$. Moreover, $\sup_{y>0} \{ \lambda(y) \}^{1/q} = \sup_{t>0} t^{1/p} h^*(t)$. Consequently the theorem of Marcinkiewicz can be stated in the following way:

Suppose $T$ is quasi-linear and, for $1 \leq p_1 \leq p_2 \leq \infty$, $i = 0, 1$, with $p_0 < p_1, q_0 \neq q_1$,

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An operator $T$ mapping functions on a measure space into functions on another measure space is called quasi-linear if $T(f+g)$ is defined whenever $Tf$ and $Tg$ are defined and $|T(f+g)(x)| \leq K( |Tf(x)| + |Tg(x)| )$ a.e., where $K$ is a positive constant independent of $f$ and $g$. 996
\( \sup_{t > 0} t^{(1/q_i)} h^*(t) \leq B_i \| f \|_{p_i}, \quad \text{for all } f \text{ in } L^{p_i}, \quad i = 0, 1, \)

where \( h = Tf \) and \( B_0, B_1 \) are independent of \( f \). Then, for \( 0 < \theta < 1 \), there exists \( B = B_\theta \) such that \( \| h \|_{q} \leq B \| f \|_{p} \) for all \( f \) in \( L^p \), \( 1/p = (1-\theta)/p_0 + \theta/p_1 \) and \( 1/q = (1-\theta)/q_0 + \theta/q_1 \).

**Proof.** Put

\[
\begin{align*}
  f^*(x) &= \begin{cases} 
  f(x) & \text{if } |f(x)| > f^*(p), \\
  0 & \text{otherwise,}
  \end{cases}
\end{align*}
\]

and \( f_t = f - f^t \), where

\[
\gamma = \frac{(1/q_0) - (1/q)}{(1/p_0) - (1/p)} = \frac{(1/q) - (1/q_0)}{(1/p) - (1/p_0)}.
\]

Then it follows easily that

\[
\begin{align*}
  f^*(y) &\leq \begin{cases} 
  f^*(y) & \text{if } 0 < y < t', \\
  0 & \text{if } y \geq t',
  \end{cases} \\
  f^*_y(y) &\leq \begin{cases} 
  f^*(y) & \text{if } 0 < y < t', \\
  f^*(t') & \text{if } y \geq t'.
  \end{cases}
\end{align*}
\]

Suppose \( p_1 < \infty \). Using (2),

\[
\begin{align*}
  \| Tf \|_{q} &= \left( \int_0^\infty \left[ t^{(1/q)}(Tf)^*(t) \right]^q \frac{dt}{t} \right)^{1/q} \\
  &\leq \left( \frac{p}{q} \right)^{(1/p) - (1/q)} \left( \int_0^\infty \left[ t^{(1/q)}(Tf)^*(t) \right]^p \frac{dt}{t} \right)^{1/p}.
\end{align*}
\]

It follows easily from the definitions that

\[
(T(f_t + f^t))^*(t) \leq 2K \left( (Tf_t)^* \left( \frac{t}{2} \right) + (Tf^t)^* \left( \frac{t}{2} \right) \right).
\]

Using this, a change of variables and Minkowski's inequality we majorize the above by

\[
(2K)^{2^{1/q}} \left( \frac{p}{q} \right)^{1/p - 1/q} \left\{ \left( \int_0^\infty \left[ t^{(1/q)}(Tf)^*(t) \right]^p \frac{dt}{t} \right)^{1/p} \\
  + \left( \int_0^\infty \left[ t^{(1/q)}(Tf)^*(t) \right]^p \frac{dt}{t} \right)^{1/p} \right\}.
\]

By (3), this is dominated by
which, by (4), (2) and Minkowski's inequality is majorized by

\[
(2K)^{2/p} \left( \frac{p}{q} \right)^{1/p-1/q} \left\{ \left( \int_0^\infty \left[ \int_0^1 y^{1/q-1/q_0} \left( y y f^*(y) \right) dy \right] p dy \right)^{1/p} dt \right\}
\]

Finally, by a change of variables and (1) the last expression is less than or equal to

\[
(2K)^{2/p} \left( \frac{p}{q} \right)^{1/p-1/q} \left\{ \left( \int_0^\infty \left[ \int_0^1 y^{1/q-1/q_0} \left( y y f^*(y) \right) dy \right] p dy \right)^{1/p} dt \right\}
\]

In case \( p_1 = q_2 = \infty \) the proof is the same except for the use of the estimate \( \| f \|_{\infty} \leq f^*(r) \).

\section*{Bibliography}


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