SHORTER NOTES

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THE MARCINKIEWICZ INTERPOLATION THEOREM[1]

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We show that the Marcinkiewicz Theorem on the interpolation of operators acting on L^p spaces (see [3, pp. 111–116]) is an immediate consequence of two easily proved inequalities. The first one is a well-known result of Hardy (see [1, pp. 245–246]):

If $q \geqq 1$, $r > 0$, *and* g *is a measurable, non-negative function on* $(0, \infty)$, *then*

$$\left(\int_0^\infty \left(\int_0^t g(y)dy\right)^q t^{-r-1}dt\right)^{1/q} \leqq \frac{q}{r}\left(\int_0^\infty (yg(y))^q y^{-r-1}dy\right)^{1/q} \text{ and}$$
$$\left(\int_0^\infty \left(\int_t^\infty g(y)dy\right)^q t^{r-1}dt\right)^{1/q} \leqq \frac{q}{r}\left(\int_0^\infty (yg(y))^q y^{r-1}dy\right)^{1/q}. \tag{1}$$

The second one can be found in [2]:

If g *is non-negative and nonincreasing on* $(0, \infty)$, $1 \leqq q_1 \leqq q_2 \leqq \infty$ *and* $1 \leqq p \leqq \infty$, *then*

$$\left(\int_0^\infty [t^{1/p}g(t)]^{q_2}\frac{dt}{t}\right)^{1/q_2} \leqq \left(\frac{q_1}{p}\right)^{1/q_1 - 1/q_2}\left(\int_0^\infty [t^{1/p}g(t)]^{q_1}\frac{dt}{t}\right)^{1/q_1}. \tag{2}$$

If h is measurable on a measure space M with measure m, its distribution function is defined for $y>0$ by $\lambda_h(y) = \lambda(y) = m\{x \in M; f(x) > y\}$.

The nonincreasing rearrangement of h onto $(0, \infty)$ is then the function given by $h^*(t) = \inf\{y>0; \lambda(y) \leqq t\}$, $t>0$. Both h^* and λ are non-negative and nonincreasing functions that are continuous from the right. h^* and h have the same distribution function, thus $\|h^*\|_p = \|h\|_p$. Moreover, $\sup_{y>0} y\{\lambda(y)\}^{1/q} = \sup_{t>0} t^{1/q}h^*(t)$. Consequently the theorem of Marcinkiewicz can be stated in the following way:

Suppose T *is quasi-linear*[2] *and, for* $1 \leqq p_i \leqq q_i \leqq \infty$, $i = 0, 1$, *with* $p_0 < p_1$, $q_0 \neq q_1$,

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[2] An operator T mapping functions on a measure space into functions on another measure space is called *quasi-linear* if $T(f+g)$ is defined whenever Tf and Tg are defined and if $|T(f+g)(x)| \leqq K(|Tf(x)| + |Tg(x)|)$ a.e., where K is a positive constant independent of f and g.

(3) $$\sup_{t>0} t^{(1/q_i)}h^*(t) \leqq B_i\|f\|_{p_i} \qquad \text{for all } f \text{ in } L^{p_i}, \qquad i = 0, 1,$$

where $h=Tf$ *and* B_0, B_1 *are independent of* f. *Then, for* $0<\theta<1$, *there exists* $B=B_\theta$ *such that* $\|h\|_q=\|Tf\|_q\leqq B\|f\|_p$ *for all* f *in* L^p, $1/p=(1-\theta)/p_0+\theta/p_1$ *and* $1/q=(1-\theta)/q_0+\theta/q_1$.

PROOF. Put

$$f^t(x) = \begin{cases} f(x) & \text{if } |f(x)| > f^*(t^\gamma), \\ 0 & \text{otherwise,} \end{cases}$$

and $f_t=f-f^t$, where

$$\gamma = \frac{(1/q_0) - (1/q)}{(1/p_0) - (1/p)} = \frac{(1/q) - (1/q_1)}{(1/p) - (1/p_1)}.$$

Then it follows easily that

(4) $$f^{t*}(y) \leqq \begin{cases} f^*(y) & \text{if } 0 < y < t^\gamma, \\ 0 & \text{if } y \geqq t^\gamma, \end{cases}$$
$$f_t^*(y) \leqq \begin{cases} f^*(t^\gamma) & \text{if } 0 < y < t^\gamma, \\ f^*(y) & \text{if } y \geqq t^\gamma. \end{cases}$$

Suppose $p_1<\infty$. Using (2),

$$\|Tf\|_q = \left(\int_0^\infty [t^{(1/q)}(Tf)^*(t)]^q \frac{dt}{t}\right)^{1/q}$$
$$\leqq \left(\frac{p}{q}\right)^{(1/p)-(1/q)}\left(\int_0^\infty [t^{(1/q)}(Tf)^*(t)]^p \frac{dt}{t}\right)^{1/p}.$$

It follows easily from the definitions that

$$(T(f_t+f^t))^*(t) \leqq 2K\left((Tf_t)^*\left(\frac{t}{2}\right) + (Tf^t)^*\left(\frac{t}{2}\right)\right).$$

Using this, a change of variables and Minkowski's inequality we majorize the above by

$$(2K)2^{1/q}\left(\frac{p}{q}\right)^{1/p-1/q}\left\{\left(\int_0^\infty [t^{1/q}(Tf^t)^*(t)]^p \frac{dt}{t}\right)^{1/p} + \left(\int_0^\infty [t^{1/q}(Tf_t)^*(t)]^p \frac{dt}{t}\right)^{1/p}\right\}.$$

By (3), this is dominated by

$$(2K)2^{1/q}\left(\frac{p}{q}\right)^{1/p-1/q}\left\{\left(\int_0^\infty\left[t^{1/q-1/q_0}\left(\int_0^\infty[y^{1/p_0}f^{t*}(y)]^{p_0}\frac{dy}{y}\right)^{1/p_0}\right]^p\frac{dt}{t}\right)^{1/p}\right.$$
$$\left.+\left(\int_0^\infty\left[t^{1/q-1/q_1}\left(\int_0^\infty[y^{1/p_1}f_t^*(y)]^{p_1}\frac{dy}{y}\right)^{1/p}\right]^p\frac{dt}{t}\right)^{1/p}\right\},$$

which, by (4), (2) and Minkowski's inequality is majorized by

$$(2K)2^{1/q}\left(\frac{p}{q}\right)^{1/p-1/q}$$
$$\cdot\left\{\left(\int_0^\infty\left[t^{1/q-1/q_0}\left(\frac{1}{p_0}\right)^{1-1/p_0}\left(\int_0^{t^\gamma}y^{1/p_0}f^*(y)\frac{dy}{y}\right)\right]^p\frac{dt}{t}\right)^{1/p}\right.$$
$$+\left(\int_0^\infty\left[t^{1/q-1/q_1}\left(\frac{1}{p_1}\right)^{1-1/p_1}\left(\int_{t^\gamma}^\infty y^{1/p_1}f^*(y)\frac{dy}{y}\right)\right]^p\frac{dt}{t}\right)^{1/p}$$
$$\left.+\left(\int_0^\infty\left[t^{1/q-1/q_1}\left(\frac{1}{p_1}\right)^{1-1/p_1}\left(\int_0^{yt^\gamma}y^{1/p_1}f^*(t^\gamma)\frac{dy}{y}\right)\right]^p\frac{dt}{t}\right)^{1/p}\right\}.$$

Finally, by a change of variables and (1) the last expression is less than or equal to

$$(2K)2^{1/q}\left(\frac{p}{q}\right)^{1/p-1/q}|\gamma|^{-1/p}$$
$$\cdot\left\{\frac{B_0\left(\frac{1}{p_0}\right)^{1-1/p_0}}{\left(\frac{1}{p_0}\right)-\left(\frac{1}{p}\right)}+\frac{B_1\left(\frac{1}{p_1}\right)^{1-1/p_1}}{\left(\frac{1}{p}\right)-\left(\frac{1}{p_1}\right)}+B_1p_1^{1/p_1}\right\}\|f\|_p=B\|f\|_p.$$

In case $p_1=q_2=\infty$ the proof is the same except for the use of the estimate $\|f_t\|_\infty\leqq f^*(t^\gamma)$.

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